NAG Library Function Document

nag_fresnel_s (s20acc)

1 Purpose

nag_fresnel_s (s20acc) returns a value for the Fresnel integral $S(x)$.

2 Specification

```
#include <nag.h>
#include <nags.h>
double nag_fresnel_s (double x)
```

3 Description

nag_fresnel_s (s20acc) evaluates an approximation to the Fresnel integral

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt.$$

Note: $S(x) = -S(-x)$, so the approximation need only consider $x \geq 0.0$.

The function is based on three Chebyshev expansions:

For $0 < x \leq 3$,

$$S(x) = x^3 \sum_{r=0} a_r T_r(t), \qquad \text{with } t = 2\left(\frac{x}{3}\right)^4 - 1.$$

For $x > 3$,

$$S(x) = \frac{1}{2} - \frac{f(x)}{x}\cos\left(\frac{\pi}{2}x^2\right) - \frac{g(x)}{x^3}\sin\left(\frac{\pi}{2}x^2\right),$$

where $f(x) = \sum_{r=0} b_r T_r(t)$,

and $g(x) = \sum_{r=0} c_r T_r(t)$,

with $t = 2\left(\frac{3}{x}\right)^4 - 1$.

For small x, $S(x) \simeq \frac{\pi}{6}x^3$. This approximation is used when x is sufficiently small for the result to be correct to ***machine precision***. For very small x, this approximation would underflow; the result is then set exactly to zero.

For large x, $f(x) \simeq \frac{1}{\pi}$ and $g(x) \simeq \frac{1}{\pi^2}$. Therefore for moderately large x, when $\frac{1}{\pi^2 x^3}$ is negligible compared with $\frac{1}{2}$, the second term in the approximation for $x > 3$ may be dropped. For very large x, when $\frac{1}{\pi x}$ becomes negligible, $S(x) \simeq \frac{1}{2}$. However there will be considerable difficulties in calculating $\cos\left(\frac{\pi}{2}x^2\right)$ accurately before this final limiting value can be used. Since $\cos\left(\frac{\pi}{2}x^2\right)$ is periodic, its value is essentially determined by the fractional part of x^2. If $x^2 = N + \theta$ where N is an integer and $0 \leq \theta < 1$, then $\cos\left(\frac{\pi}{2}x^2\right)$ depends on θ and on N modulo 4. By exploiting this fact, it is possible to retain significance in the calculation of $\cos\left(\frac{\pi}{2}x^2\right)$ either all the way to the very large x limit, or at least until the integer part of $\frac{x}{2}$ is equal to the maximum integer allowed on the machine.

4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

5 Arguments

1: **x** – double *Input*

On entry: the argument x of the function.

6 Error Indicators and Warnings

None.

7 Accuracy

Let δ and ϵ be the relative errors in the argument and result respectively.

If δ is somewhat larger than the ***machine precision*** (i.e., if δ is due to data errors etc.), then ϵ and δ are approximately related by:

$$\epsilon \simeq \left|\frac{x\sin\left(\frac{\pi}{2}x^2\right)}{S(x)}\right|\delta.$$

Figure 1 shows the behaviour of the error amplification factor $\left|\frac{x\sin\left(\frac{\pi}{2}x^2\right)}{S(x)}\right|$.

However if δ is of the same order as the ***machine precision***, then rounding errors could make ϵ slightly larger than the above relation predicts.

For small x, $\epsilon \simeq 3\delta$ and hence there is only moderate amplification of relative error. Of course for very small x where the correct result would underflow and exact zero is returned, relative error-control is lost.

For moderately large values of x,

$$|\epsilon| \simeq \left|2x\sin\left(\frac{\pi}{2}x^2\right)\right||\delta|$$

and the result will be subject to increasingly large amplification of errors. However the above relation breaks down for large values of x (i.e., when $\frac{1}{x^2}$ is of the order of the ***machine precision***); in this region the relative error in the result is essentially bounded by $\frac{2}{\pi x}$.

Hence the effects of error amplification are limited and at worst the relative error loss should not exceed half the possible number of significant figures.

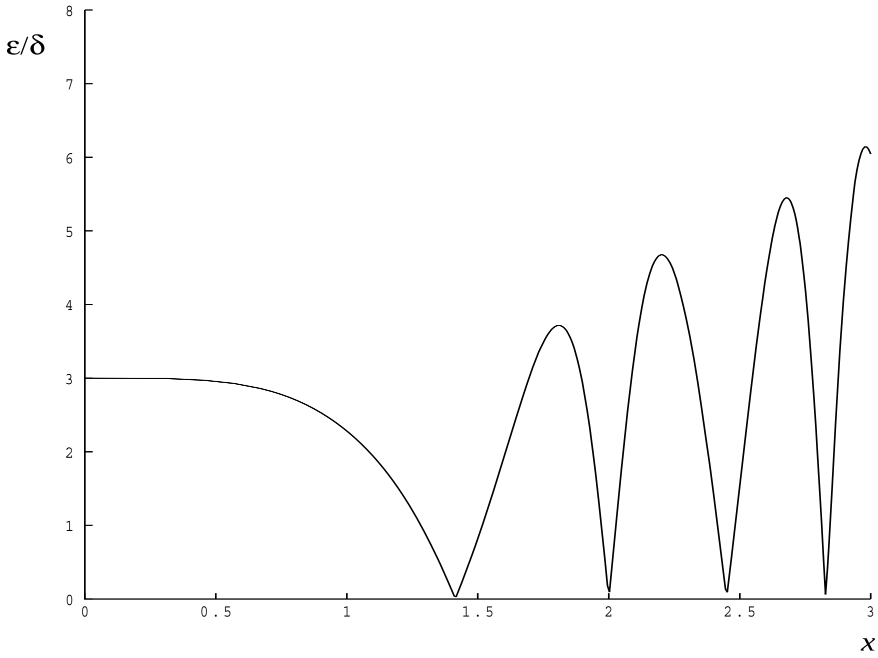


Figure 1

8 Parallelism and Performance

nag_fresnel_s (s20acc) is not threaded in any implementation.

9 Further Comments

None.

10 Example

This example reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

10.1 Program Text

```
/* nag_fresnel_s (s20acc) Example Program.
 *
 * NAGPRODCODE Version.
 *
 * Copyright 2016 Numerical Algorithms Group.
 *
 * Mark 26, 2016.
 */

#include <nag.h>
#include <stdio.h>
#include <nag_stdlib.h>
#include <nags.h>

int main(void)
{
  Integer exit_status = 0;
  double x, y;

  /* Skip heading in data file */
#ifdef _WIN32
  scanf_s("%*[^\n]");
#else
```

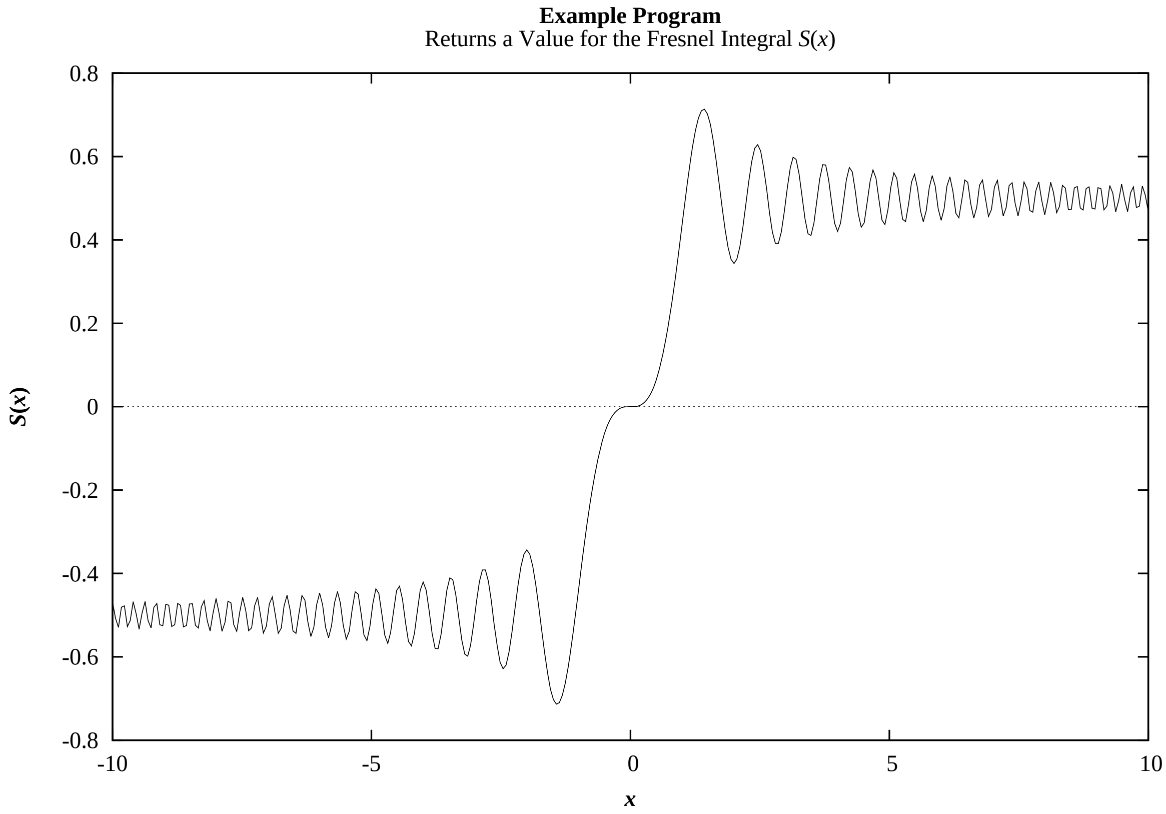
```
  scanf("%*[^\n]");
#endif
  printf("nag_fresnel_s (s20acc) Example Program Results\n");
  printf("     x          y\n");
#ifdef _WIN32
  while (scanf_s("%lf", &x) != EOF)
#else
  while (scanf("%lf", &x) != EOF)
#endif
  {
    /* nag_fresnel_s (s20acc).
     * Fresnel integral S(x)
     */
    y = nag_fresnel_s(x);
    printf("%12.3e%12.3e\n", x, y);
  }

  return exit_status;
}
```

10.2 Program Data

```
nag_fresnel_s (s20acc) Example Program Data
             0.0
             0.5
             1.0
             2.0
             4.0
             5.0
             6.0
             8.0
             10.0
             -1.0
             1000.0
```

10.3 Program Results

```
nag_fresnel_s (s20acc) Example Program Results
     x          y
   0.000e+00   0.000e+00
   5.000e-01   6.473e-02
   1.000e+00   4.383e-01
   2.000e+00   3.434e-01
   4.000e+00   4.205e-01
   5.000e+00   4.992e-01
   6.000e+00   4.470e-01
   8.000e+00   4.602e-01
   1.000e+01   4.682e-01
  -1.000e+00  -4.383e-01
   1.000e+03   4.997e-01
```



Example Program
Returns a Value for the Fresnel Integral *S*(*x*)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
S(*x*)
-10
-5
0
5
10
x