

# NAG Library Routine Document

## G08CDF

**Note:** before using this routine, please read the Users' Note for your implementation to check the interpretation of *bold italicised* terms and other implementation-dependent details.

### 1 Purpose

G08CDF performs the two sample Kolmogorov–Smirnov distribution test.

### 2 Specification

```
SUBROUTINE G08CDF (N1, X, N2, Y, NTYPE, D, Z, P, SX, SY, IFAIL)
INTEGER          N1, N2, NTYPE, IFAIL
REAL (KIND=nag_wp) X(N1), Y(N2), D, Z, P, SX(N1), SY(N2)
```

### 3 Description

The data consists of two independent samples, one of size  $n_1$ , denoted by  $x_1, x_2, \dots, x_{n_1}$ , and the other of size  $n_2$  denoted by  $y_1, y_2, \dots, y_{n_2}$ . Let  $F(x)$  and  $G(x)$  represent their respective, unknown, distribution functions. Also let  $S_1(x)$  and  $S_2(x)$  denote the values of the sample cumulative distribution functions at the point  $x$  for the two samples respectively.

The Kolmogorov–Smirnov test provides a test of the null hypothesis  $H_0: F(x) = G(x)$  against one of the following alternative hypotheses:

- (i)  $H_1: F(x) \neq G(x)$ .
- (ii)  $H_2: F(x) > G(x)$ . This alternative hypothesis is sometimes stated as, ‘The  $x$ 's tend to be smaller than the  $y$ 's’, i.e., it would be demonstrated in practical terms if the values of  $S_1(x)$  tended to exceed the corresponding values of  $S_2(x)$ .
- (iii)  $H_3: F(x) < G(x)$ . This alternative hypothesis is sometimes stated as, ‘The  $x$ 's tend to be larger than the  $y$ 's’, i.e., it would be demonstrated in practical terms if the values of  $S_2(x)$  tended to exceed the corresponding values of  $S_1(x)$ .

One of the following test statistics is computed depending on the particular alternative null hypothesis specified (see the description of the argument NTYPE in Section 5).

For the alternative hypothesis  $H_1$ .

$D_{n_1, n_2}$  – the largest absolute deviation between the two sample cumulative distribution functions.

For the alternative hypothesis  $H_2$ .

$D_{n_1, n_2}^+$  – the largest positive deviation between the sample cumulative distribution function of the first sample,  $S_1(x)$ , and the sample cumulative distribution function of the second sample,  $S_2(x)$ .  
Formally  $D_{n_1, n_2}^+ = \max\{S_1(x) - S_2(x), 0\}$ .

For the alternative hypothesis  $H_3$ .

$D_{n_1, n_2}^-$  – the largest positive deviation between the sample cumulative distribution function of the second sample,  $S_2(x)$ , and the sample cumulative distribution function of the first sample,  $S_1(x)$ .  
Formally  $D_{n_1, n_2}^- = \max\{S_2(x) - S_1(x), 0\}$ .

G08CDF also returns the standardized statistic  $Z = \sqrt{\frac{n_1 + n_2}{n_1 n_2}} \times D$ , where  $D$  may be  $D_{n_1, n_2}$ ,  $D_{n_1, n_2}^+$  or  $D_{n_1, n_2}^-$  depending on the choice of the alternative hypothesis. The distribution of this statistic converges asymptotically to a distribution given by Smirnov as  $n_1$  and  $n_2$  increase; see Feller (1948), Kendall and Stuart (1973), Kim and Jenrich (1973), Smirnov (1933) or Smirnov (1948).

The probability, under the null hypothesis, of obtaining a value of the test statistic as extreme as that observed, is computed. If  $\max(n_1, n_2) \leq 2500$  and  $n_1 n_2 \leq 10000$  then an exact method given by Kim and Jenrich (see Kim and Jenrich (1973)) is used. Otherwise  $p$  is computed using the approximations suggested by Kim and Jenrich (1973). Note that the method used is only exact for continuous theoretical distributions. This method computes the two-sided probability. The one-sided probabilities are estimated by halving the two-sided probability. This is a good estimate for small  $p$ , that is  $p \leq 0.10$ , but it becomes very poor for larger  $p$ .

## 4 References

Conover W J (1980) *Practical Nonparametric Statistics* Wiley

Feller W (1948) On the Kolmogorov–Smirnov limit theorems for empirical distributions *Ann. Math. Statist.* **19** 179–181

Kendall M G and Stuart A (1973) *The Advanced Theory of Statistics (Volume 2)* (3rd Edition) Griffin

Kim P J and Jenrich R I (1973) Tables of exact sampling distribution of the two sample Kolmogorov–Smirnov criterion  $D_{mn}(m < n)$  *Selected Tables in Mathematical Statistics* **1** 80–129 American Mathematical Society

Siegel S (1956) *Non-parametric Statistics for the Behavioral Sciences* McGraw–Hill

Smirnov N (1933) Estimate of deviation between empirical distribution functions in two independent samples *Bull. Moscow Univ.* **2(2)** 3–16

Smirnov N (1948) Table for estimating the goodness of fit of empirical distributions *Ann. Math. Statist.* **19** 279–281

## 5 Arguments

- 1: N1 – INTEGER *Input*  
*On entry:* the number of observations in the first sample,  $n_1$ .  
*Constraint:*  $N1 \geq 1$ .
- 2: X(N1) – REAL (KIND=nag\_wp) array *Input*  
*On entry:* the observations from the first sample,  $x_1, x_2, \dots, x_{n_1}$ .
- 3: N2 – INTEGER *Input*  
*On entry:* the number of observations in the second sample,  $n_2$ .  
*Constraint:*  $N2 \geq 1$ .
- 4: Y(N2) – REAL (KIND=nag\_wp) array *Input*  
*On entry:* the observations from the second sample,  $y_1, y_2, \dots, y_{n_2}$ .
- 5: NTYPE – INTEGER *Input*  
*On entry:* the statistic to be computed, i.e., the choice of alternative hypothesis.  
 NTYPE = 1  
     Computes  $D_{n_1 n_2}$ , to test against  $H_1$ .  
 NTYPE = 2  
     Computes  $D_{n_1 n_2}^+$ , to test against  $H_2$ .  
 NTYPE = 3  
     Computes  $D_{n_1 n_2}^-$ , to test against  $H_3$ .  
*Constraint:* NTYPE = 1, 2 or 3.

- 6: D – REAL (KIND=nag\_wp) Output  
*On exit:* the Kolmogorov–Smirnov test statistic ( $D_{n_1n_2}$ ,  $D_{n_1n_2}^+$  or  $D_{n_1n_2}^-$  according to the value of NTYPE).
- 7: Z – REAL (KIND=nag\_wp) Output  
*On exit:* a standardized value,  $Z$ , of the test statistic,  $D$ , without any correction for continuity.
- 8: P – REAL (KIND=nag\_wp) Output  
*On exit:* the tail probability associated with the observed value of  $D$ , where  $D$  may be  $D_{n_1n_2}$ ,  $D_{n_1n_2}^+$  or  $D_{n_1n_2}^-$  depending on the value of NTYPE (see Section 3).
- 9: SX(N1) – REAL (KIND=nag\_wp) array Output  
*On exit:* the observations from the first sample sorted in ascending order.
- 10: SY(N2) – REAL (KIND=nag\_wp) array Output  
*On exit:* the observations from the second sample sorted in ascending order.
- 11: IFAIL – INTEGER Input/Output  
*On entry:* IFAIL must be set to 0, –1 or 1. If you are unfamiliar with this argument you should refer to Section 3.4 in How to Use the NAG Library and its Documentation for details.  
 For environments where it might be inappropriate to halt program execution when an error is detected, the value –1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this argument, the recommended value is 0. **When the value –1 or 1 is used it is essential to test the value of IFAIL on exit.**  
*On exit:* IFAIL = 0 unless the routine detects an error or a warning has been flagged (see Section 6).

## 6 Error Indicators and Warnings

If on entry IFAIL = 0 or –1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

On entry,  $N1 < 1$ ,  
 or  $N2 < 1$ .

IFAIL = 2

On entry, NTYPE  $\neq$  1, 2 or 3.

IFAIL = 3

The iterative procedure used in the approximation of the probability for large  $n_1$  and  $n_2$  did not converge. For the two-sided test,  $p = 1$  is returned. For the one-sided test,  $p = 0.5$  is returned.

IFAIL = –99

An unexpected error has been triggered by this routine. Please contact NAG.

See Section 3.9 in How to Use the NAG Library and its Documentation for further information.

IFAIL = -399

Your licence key may have expired or may not have been installed correctly.

See Section 3.8 in How to Use the NAG Library and its Documentation for further information.

IFAIL = -999

Dynamic memory allocation failed.

See Section 3.7 in How to Use the NAG Library and its Documentation for further information.

## 7 Accuracy

The large sample distributions used as approximations to the exact distribution should have a relative error of less than 5% for most cases.

## 8 Parallelism and Performance

G08CDF is threaded by NAG for parallel execution in multithreaded implementations of the NAG Library.

Please consult the X06 Chapter Introduction for information on how to control and interrogate the OpenMP environment used within this routine. Please also consult the Users' Note for your implementation for any additional implementation-specific information.

## 9 Further Comments

The time taken by G08CDF increases with  $n_1$  and  $n_2$ , until  $n_1 n_2 > 10000$  or  $\max(n_1, n_2) \geq 2500$ . At this point one of the approximations is used and the time decreases significantly. The time then increases again modestly with  $n_1$  and  $n_2$ .

## 10 Example

This example computes the two-sided Kolmogorov–Smirnov test statistic for two independent samples of size 100 and 50 respectively. The first sample is from a uniform distribution  $U(0, 2)$ . The second sample is from a uniform distribution  $U(0.25, 2.25)$ . The test statistic,  $D_{n_1, n_2}$ , the standardized test statistic,  $Z$ , and the tail probability,  $p$ , are computed and printed.

### 10.1 Program Text

```

Program g08cdfe

!      G08CDF Example Program Text

!      Mark 26 Release. NAG Copyright 2016.

!      .. Use Statements ..
      Use nag_library, Only: g08cdf, nag_wp
!      .. Implicit None Statement ..
      Implicit None
!      .. Parameters ..
      Integer, Parameter          :: nin = 5, nout = 6
!      .. Local Scalars ..
      Real (Kind=nag_wp)          :: d, p, z
      Integer                     :: ifail, n1, n2, ntype
!      .. Local Arrays ..
      Real (Kind=nag_wp), Allocatable :: sx(:), sy(:), x(:), y(:)
!      .. Executable Statements ..
      Write (nout,*) 'G08CDF Example Program Results'
      Write (nout,*)

!      Skip heading in data file
      Read (nin,*)

```

```

!      Read in the problem size and statistic type
      Read (nin,*) n1, n2, ntype

      Allocate (x(n1),y(n2),sx(n1),sy(n2))

!      Read in data
      Read (nin,*) x(1:n1)
      Read (nin,*) y(1:n2)

!      Perform test
      ifail = 0
      Call g08cdf(n1,x,n2,y,ntype,d,z,p,sx,sy,ifail)

!      Display results
      Write (nout,99999) 'Test statistic D = ', d
      Write (nout,99999) 'Z statistic      = ', z
      Write (nout,99999) 'Tail probability = ', p

99999 Format (1X,A,F8.4)
      End Program g08cdf

```

## 10.2 Program Data

G08CDF Example Program Data

```

100 50 1  :: NX,NY,NTYPE
1.160 1.785 0.322 1.437 1.695 1.770 1.209 0.479 1.122 0.974
0.290 1.155 0.218 1.595 1.053 1.058 1.282 1.278 1.066 0.725
0.113 1.516 1.329 1.907 0.101 0.387 1.392 0.613 0.692 1.397
1.627 0.417 1.079 0.607 0.899 0.493 0.381 1.660 0.233 0.718
1.376 1.395 1.557 1.610 1.632 0.851 1.824 0.921 0.139 0.618
0.050 0.956 0.669 1.109 1.882 1.462 1.465 0.201 1.036 1.127
0.907 0.876 1.199 1.667 1.141 0.820 0.488 0.732 0.725 0.753
0.760 1.833 0.074 1.101 0.620 1.858 0.681 0.705 0.876 1.096
1.870 1.597 0.990 0.430 0.410 0.399 1.693 0.492 1.318 0.883
1.291 1.051 1.934 1.314 1.496 0.391 1.079 0.881 0.983 1.306 :: End of X
1.695 1.452 0.997 1.771 1.114 1.624 2.005 0.782 1.870 0.954
1.606 2.059 0.774 0.741 1.040 0.521 2.163 0.818 1.781 1.420
0.558 1.437 2.004 1.325 0.398 0.582 2.047 0.332 1.186 0.890
1.825 1.324 1.334 0.261 0.299 1.733 1.172 1.000 1.663 1.093
1.045 2.022 1.174 0.670 1.143 1.189 0.494 1.275 1.122 1.823 :: End of Y

```

## 10.3 Program Results

G08CDF Example Program Results

```

Test statistic D = 0.1800
Z statistic      = 0.0312
Tail probability = 0.2222

```

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