

# NAG Library Routine Document

## F01KCF

**Note:** before using this routine, please read the Users' Note for your implementation to check the interpretation of *bold italicised* terms and other implementation-dependent details.

### 1 Purpose

F01KCF computes an estimate of the absolute condition number of a matrix function  $f$  of a complex  $n$  by  $n$  matrix  $A$  in the 1-norm, using analytical derivatives of  $f$  you have supplied.

### 2 Specification

SUBROUTINE F01KCF (N, A, LDA, F, IUSER, RUSER, IFLAG, CONDA, NORMA, &  
NORMFA, IFAIL)

INTEGER N, LDA, IUSER(\*), IFLAG, IFAIL  
REAL (KIND=nag\_wp) RUSER(\*), CONDA, NORMA, NORMFA  
COMPLEX (KIND=nag\_wp) A(LDA,\*)  
EXTERNAL F

### 3 Description

The absolute condition number of  $f$  at  $A$ ,  $\text{cond}_{\text{abs}}(f, A)$  is given by the norm of the Fréchet derivative of  $f$ ,  $L(A)$ , which is defined by

$$\|L(X)\| := \max_{E \neq 0} \frac{\|L(X, E)\|}{\|E\|},$$

where  $L(X, E)$  is the Fréchet derivative in the direction  $E$ .  $L(X, E)$  is linear in  $E$  and can therefore be written as

$$\text{vec}(L(X, E)) = K(X)\text{vec}(E),$$

where the  $\text{vec}$  operator stacks the columns of a matrix into one vector, so that  $K(X)$  is  $n^2 \times n^2$ . F01KCF computes an estimate  $\gamma$  such that  $\gamma \leq \|K(X)\|_1$ , where  $\|K(X)\|_1 \in [n^{-1}\|L(X)\|_1, n\|L(X)\|_1]$ . The relative condition number can then be computed via

$$\text{cond}_{\text{rel}}(f, A) = \frac{\text{cond}_{\text{abs}}(f, A)\|A\|_1}{\|f(A)\|_1}.$$

The algorithm used to find  $\gamma$  is detailed in Section 3.4 of Higham (2008).

### 4 References

Higham N J (2008) *Functions of Matrices: Theory and Computation* SIAM, Philadelphia, PA, USA

### 5 Arguments

- 1: N – INTEGER *Input*  
*On entry:*  $n$ , the order of the matrix  $A$ .  
*Constraint:*  $N \geq 0$ .
- 2: A(LDA,\*) – COMPLEX (KIND=nag\_wp) array *Input/Output*  
**Note:** the second dimension of the array  $A$  must be at least  $N$ .  
*On entry:* the  $n$  by  $n$  matrix  $A$ .

*On exit:* the  $n$  by  $n$  matrix,  $f(A)$ .

3: LDA – INTEGER *Input*

*On entry:* the first dimension of the array A as declared in the (sub)program from which F01KCF is called.

*Constraint:*  $LDA \geq N$ .

4: F – SUBROUTINE, supplied by the user. *External Procedure*

Given an integer  $m$ , the subroutine F evaluates  $f^{(m)}(z_i)$  at a number of points  $z_i$ .

The specification of F is:

```
SUBROUTINE F (M, IFLAG, NZ, Z, FZ, IUSER, RUSER)
INTEGER                M, IFLAG, NZ, IUSER(*)
REAL (KIND=nag_wp)    RUSER(*)
COMPLEX (KIND=nag_wp) Z(NZ), FZ(NZ)
```

1: M – INTEGER *Input*

*On entry:* the order,  $m$ , of the derivative required.

If  $M = 0$ ,  $f(z_i)$  should be returned. For  $M > 0$ ,  $f^{(m)}(z_i)$  should be returned.

2: IFLAG – INTEGER *Input/Output*

*On entry:* IFLAG will be zero.

*On exit:* IFLAG should either be unchanged from its entry value of zero, or may be set nonzero to indicate that there is a problem in evaluating the function  $f(z)$ ; for instance  $f(z)$  may not be defined. If IFLAG is returned as nonzero then F01KCF will terminate the computation, with IFAIL = 3.

3: NZ – INTEGER *Input*

*On entry:*  $n_z$ , the number of function or derivative values required.

4: Z(NZ) – COMPLEX (KIND=nag\_wp) array *Input*

*On entry:* the  $n_z$  points  $z_1, z_2, \dots, z_{n_z}$  at which the function  $f$  is to be evaluated.

5: FZ(NZ) – COMPLEX (KIND=nag\_wp) array *Output*

*On exit:* the  $n_z$  function or derivative values.  $FZ(i)$  should return the value  $f^{(m)}(z_i)$ , for  $i = 1, 2, \dots, n_z$ .

6: IUSER(\*) – INTEGER array *User Workspace*

7: RUSER(\*) – REAL (KIND=nag\_wp) array *User Workspace*

F is called with the arguments IUSER and RUSER as supplied to F01KCF. You should use the arrays IUSER and RUSER to supply information to F.

F must either be a module subprogram USED by, or declared as EXTERNAL in, the (sub) program from which F01KCF is called. Arguments denoted as *Input* must **not** be changed by this procedure.

5: IUSER(\*) – INTEGER array *User Workspace*

6: RUSER(\*) – REAL (KIND=nag\_wp) array *User Workspace*

IUSER and RUSER are not used by F01KCF, but are passed directly to F and should be used to pass information to this routine.

- 7: IFLAG – INTEGER *Output*  
*On exit:* IFLAG = 0, unless IFLAG has been set nonzero inside F, in which case IFLAG will be the value set and IFAIL will be set to IFAIL = 3.
- 8: CONDA – REAL (KIND=nag\_wp) *Output*  
*On exit:* an estimate of the absolute condition number of  $f$  at  $A$ .
- 9: NORMA – REAL (KIND=nag\_wp) *Output*  
*On exit:* the 1-norm of  $A$ .
- 10: NORMFA – REAL (KIND=nag\_wp) *Output*  
*On exit:* the 1-norm of  $f(A)$ .
- 11: IFAIL – INTEGER *Input/Output*  
*On entry:* IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this argument you should refer to Section 3.4 in How to Use the NAG Library and its Documentation for details.  
 For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this argument, the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**  
*On exit:* IFAIL = 0 unless the routine detects an error or a warning has been flagged (see Section 6).

## 6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

An internal error occurred when estimating the norm of the Fréchet derivative of  $f$  at  $A$ . Please contact NAG.

IFAIL = 2

An internal error occurred when evaluating the matrix function  $f(A)$ . You can investigate further by calling F01FMF with the matrix  $A$  and the function  $f$ .

IFAIL = 3

IFLAG has been set nonzero by the user-supplied subroutine.

IFAIL = -1

On entry,  $N < 0$ .

IFAIL = -3

On entry, argument LDA is invalid.  
 Constraint:  $LDA \geq N$ .

IFAIL = -99

An unexpected error has been triggered by this routine. Please contact NAG.

See Section 3.9 in How to Use the NAG Library and its Documentation for further information.

IFAIL = -399

Your licence key may have expired or may not have been installed correctly.

See Section 3.8 in How to Use the NAG Library and its Documentation for further information.

IFAIL = -999

Dynamic memory allocation failed.

See Section 3.7 in How to Use the NAG Library and its Documentation for further information.

## 7 Accuracy

F01KCF uses the norm estimation routine F04ZDF to estimate a quantity  $\gamma$ , where  $\gamma \leq \|K(X)\|_1$  and  $\|K(X)\|_1 \in [n^{-1}\|L(X)\|_1, n\|L(X)\|_1]$ . For further details on the accuracy of norm estimation, see the documentation for F04ZDF.

## 8 Parallelism and Performance

F01KCF is threaded by NAG for parallel execution in multithreaded implementations of the NAG Library. In these implementations, this routine may make calls to the user-supplied functions from within an OpenMP parallel region. Thus OpenMP directives within the user functions can only be used if you are compiling the user-supplied function and linking the executable in accordance with the instructions in the Users' Note for your implementation. The user workspace arrays IUSER and RUSER are classified as OpenMP shared memory and use of IUSER and RUSER has to take account of this in order to preserve thread safety whenever information is written back to either of these arrays. If at all possible, it is recommended that these arrays are only used to supply read-only data to the user functions when a multithreaded implementation is being used.

F01KCF makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the X06 Chapter Introduction for information on how to control and interrogate the OpenMP environment used within this routine. Please also consult the Users' Note for your implementation for any additional implementation-specific information.

## 9 Further Comments

Approximately  $6n^2$  of complex allocatable memory is required by the routine, in addition to the memory used by the underlying matrix function routine F01FMF.

F01KCF returns the matrix function  $f(A)$ . This is computed using F01FMF. If only  $f(A)$  is required, without an estimate of the condition number, then it is far more efficient to use F01FMF directly.

The real analogue of this routine is F01JCF.

## 10 Example

This example estimates the absolute and relative condition numbers of the matrix function  $e^{3A}$  where

$$A = \begin{pmatrix} 1.0 + 1.0i & 0.0 + 1.0i & 1.0 + 0.0i & 2.0 + 0.0i \\ 0.0 + 0.0i & 2.0 + 0.0i & 0.0 + 2.0i & 1.0 + 0.0i \\ 0.0 + 1.0i & 0.0 + 1.0i & 0.0 + 0.0i & 2.0 + 0.0i \\ 1.0 + 0.0i & 0.0 + 1.0i & 1.0 + 0.0i & 0.0 + 1.0i \end{pmatrix}.$$

## 10.1 Program Text

```

!   F01KCF Example Program Text
!   Mark 26 Release. NAG Copyright 2016.

Module f01kcfe_mod

!   .. Use Statements ..
Use nag_library, Only: nag_wp
!   .. Implicit None Statement ..
Implicit None
!   .. Accessibility Statements ..
Private
Public                               :: fexp3
Contains
  Subroutine fexp3(m,iflag,nz,z,fz,iuser,ruser)

!   .. Scalar Arguments ..
Integer, Intent (Inout)             :: iflag
Integer, Intent (In)                 :: m, nz
!   .. Array Arguments ..
Complex (Kind=nag_wp), Intent (Out) :: fz(nz)
Complex (Kind=nag_wp), Intent (In)  :: z(nz)
Real (Kind=nag_wp), Intent (Inout)  :: ruser(*)
Integer, Intent (Inout)              :: iuser(*)
!   .. Intrinsic Procedures ..
Intrinsic                             :: cmplx, exp
!   .. Executable Statements ..
Continue
  fz(1:nz) = (cmplx(3.0E0_nag_wp,0.0_nag_wp,kind=nag_wp)**m)*      &
    exp((3.0E0_nag_wp,0.0E0_nag_wp)*z(1:nz))
!   Set iflag nonzero to terminate execution for any reason.
  iflag = 0
  Return
End Subroutine fexp3
End Module f01kcfe_mod
Program f01kcfe

!   F01KCF Example Main Program

!   .. Use Statements ..
Use nag_library, Only: f01kcf, nag_wp, x02ajf, x04daf
Use f01kcfe_mod, Only: fexp3
!   .. Implicit None Statement ..
Implicit None
!   .. Parameters ..
Integer, Parameter                    :: nin = 5, nout = 6
!   .. Local Scalars ..
Real (Kind=nag_wp)                    :: conda, cond_rel, eps, norma, normfa
Integer                                :: i, ifail, iflag, lda, n
!   .. Local Arrays ..
Complex (Kind=nag_wp), Allocatable :: a(:, :)
Real (Kind=nag_wp)                   :: ruser(1)
Integer                                :: iuser(1)
!   .. Executable Statements ..
Write (nout,*) 'F01KCF Example Program Results'
Write (nout,*)
Flush (nout)

!   Skip heading in data file
Read (nin,*)
Read (nin,*) n

  lda = n
  Allocate (a(lda,n))

!   Read A from data file
Read (nin,*)(a(i,1:n),i=1,n)

!   Display A
ifail = 0

```

```

      Call x04daf('G','N',n,n,a,lda,'A',ifail)

!      Find absolute condition number estimate
      ifail = 0
      Call f01kcf(n,a,lda,fexp3,iuser,ruser,iflag,conda,norma,normfa,ifail)

      If (ifail==0) Then
!      Print solution
      Write (nout,*)
      Write (nout,*) 'F(A) = exp(3A)'
      Write (nout,99999) 'Estimated absolute condition number is: ', conda

!      Find relative condition number estimate
      eps = x02ajf()
      If (normfa>eps) Then
        cond_rel = conda*norma/normfa
        Write (nout,99999) 'Estimated relative condition number is: ',      &
          cond_rel
      Else
        Write (nout,99998) 'The estimated norm of f(A) is effectively zero', &
          'and so the relative condition number is undefined.'
      End If

      End If

99999 Format (1X,A,F7.2)
99998 Format (/ ,1X,A,/ ,1X,A)
      End Program f01kcf

```

## 10.2 Program Data

F01KCF Example Program Data

```

4                                     :Value of N

(1.0, 1.0)   (0.0, 1.0)   (1.0, 0.0)   (2.0, 0.0)
(0.0, 0.0)   (2.0, 0.0)   (0.0, 2.0)   (1.0, 0.0)
(0.0, 1.0)   (0.0, 1.0)   (0.0, 0.0)   (2.0, 0.0)
(1.0, 0.0)   (0.0, 1.0)   (1.0, 0.0)   (0.0, 1.0) :End of matrix A

```

## 10.3 Program Results

F01KCF Example Program Results

```

A
      1      2      3      4
1      1.0000  0.0000  1.0000  2.0000
      1.0000  1.0000  0.0000  0.0000

2      0.0000  2.0000  0.0000  1.0000
      0.0000  0.0000  2.0000  0.0000

3      0.0000  0.0000  0.0000  2.0000
      1.0000  1.0000  0.0000  0.0000

4      1.0000  0.0000  1.0000  0.0000
      0.0000  1.0000  0.0000  1.0000

```

```

F(A) = exp(3A)
Estimated absolute condition number is: 9474.43
Estimated relative condition number is: 13.74

```

---