NAG Library Routine Document

F04FFF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of **bold italicised** terms and other implementation-dependent details.

1 Purpose

F04FFF solves the equations $Tx = b$, where T is a real symmetric positive definite Toeplitz matrix.

2 Specification

```
SUBROUTINE F04FFF (N, T, B, X, WANTP, P, WORK, IFAIL)

INTEGER          N, IFAIL
REAL (KIND=nag_wp) T(0:*), B(*), X(N), P(*), WORK(2*(N-1))
LOGICAL          WANTP
```

3 Description

F04FFF solves the equations

$$Tx = b,$$

where T is the n by n symmetric positive definite Toeplitz matrix

$$T = \begin{pmatrix} \tau_0 & \tau_1 & \tau_2 & \cdots & \tau_{n-1} \\ \tau_1 & \tau_0 & \tau_1 & \cdots & \tau_{n-2} \\ \tau_2 & \tau_1 & \tau_0 & \cdots & \tau_{n-3} \\ . & . & . & & . \\ \tau_{n-1} & \tau_{n-2} & \tau_{n-3} & \cdots & \tau_0 \end{pmatrix}$$

and b is an n-element vector.

The routine uses the method of Levinson (see Levinson (1947) and Golub and Van Loan (1996)). Optionally, the reflection coefficients for each step may also be returned.

4 References

Bunch J R (1985) Stability of methods for solving Toeplitz systems of equations *SIAM J. Sci. Statist. Comput.* **6** 349–364

Bunch J R (1987) The weak and strong stability of algorithms in numerical linear algebra *Linear Algebra Appl.* **88/89** 49–66

Cybenko G (1980) The numerical stability of the Levinson–Durbin algorithm for Toeplitz systems of equations *SIAM J. Sci. Statist. Comput.* **1** 303–319

Golub G H and Van Loan C F (1996) *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Levinson N (1947) The Weiner RMS error criterion in filter design and prediction *J. Math. Phys.* **25** 261–278

5 Parameters

1:    N – INTEGER   *Input*

On entry: the order of the Toeplitz matrix T.

Constraint: $N \geq 0$. When $N = 0$, then an immediate return is effected.

2:    $T(0 : *)$ – REAL (KIND=nag_wp) array   *Input*

Note: the dimension of the array T must be at least $\max(1, N)$.

On entry: $T(i)$ must contain the value τ_i, for $i = 0, 1, \ldots, N - 1$.

Constraint: $T(0) > 0.0$. Note that if this is not true, then the Toeplitz matrix cannot be positive definite.

3:    $B(*)$ – REAL (KIND=nag_wp) array   *Input*

Note: the dimension of the array B must be at least $\max(1, N)$.

On entry: the right-hand side vector b.

4:    $X(N)$ – REAL (KIND=nag_wp) array   *Output*

On exit: the solution vector x.

5:    WANTP – LOGICAL   *Input*

On entry: must be set to .TRUE. if the reflection coefficients are required, and must be set to .FALSE. otherwise.

6:    $P(*)$ – REAL (KIND=nag_wp) array   *Output*

Note: the dimension of the array P must be at least $\max(1, N - 1)$ if WANTP $=$ .TRUE., and at least 1 otherwise.

On exit: with WANTP as .TRUE., the ith element of P contains the reflection coefficient, p_i, for the ith step, for $i = 1, 2, \ldots, N - 1$. (See Section 8.) If WANTP is .FALSE., then P is not referenced.

7:    $WORK(2 \times (N - 1))$ – REAL (KIND=nag_wp) array   *Workspace*

8:    IFAIL – INTEGER   *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this parameter you should refer to Section 3.3 in the Essential Introduction for details.

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, because for this routine the values of the output parameters may be useful even if IFAIL $\neq 0$ on exit, the recommended value is -1. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

On exit: IFAIL $= 0$ unless the routine detects an error or a warning has been flagged (see Section 6).

6 Error Indicators and Warnings

If on entry IFAIL $= 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Note: F04FFF may return useful information for one or more of the following detected errors or warnings.

Errors or warnings detected by the routine:

IFAIL $= -1$

On entry, $N < 0$,
or $\quad\quad$ T(0) ≤ 0.0.

IFAIL > 0

The principal minor of order IFAIL of the Toeplitz matrix is not positive definite to working accuracy. The first (IFAIL -1) elements of X return the solution of the equations

$$T_{\text{IFAIL}-1}x = (b_1, b_2, \ldots, b_{\text{IFAIL}-1})^{\text{T}},$$

where T_k is the kth principal minor of T.

7 Accuracy

The computed solution of the equations certainly satisfies

$$r = Tx - b,$$

where $\|r\|$ is approximately bounded by

$$\|r\| \leq c\epsilon C(T),$$

c being a modest function of n, ϵ being the **machine precision** and $C(T)$ being the condition number of T with respect to inversion. This bound is almost certainly pessimistic, but it seems unlikely that the method of Levinson is backward stable, so caution should be exercised when T is ill-conditioned. The following bound on T^{-1} holds:

$$\max\left(\frac{1}{\displaystyle\prod_{i=1}^{n-1}\left(1 - p_i^2\right)}, \frac{1}{\displaystyle\prod_{i=1}^{n-1}\left(1 - p_i\right)}\right) \leq \|T^{-1}\|_1 \leq \prod_{i=1}^{n-1}\left(\frac{1 + |p_i|}{1 - |p_i|}\right).$$

(See Golub and Van Loan (1996).) The norm of T^{-1} may also be estimated using routine F04YDF. For further information on stability issues see Bunch (1985), Bunch (1987), Cybenko (1980) and Golub and Van Loan (1996).

8 Further Comments

The number of floating point operations used by F04FFF is approximately $4n^2$.

If y_i is the solution of the equations

$$T_i y_i = -(\tau_1 \tau_2 \ldots \tau_i)^{\text{T}},$$

then the partial correlation coefficient p_i is defined as the ith element of y_i.

9 Example

This example finds the solution of the equations $Tx = b$, where

$$T = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

9.1 Program Text

```
      Program f04fffe

!     F04FFF Example Program Text

!     Mark 24 Release. NAG Copyright 2012.

!     .. Use Statements ..
      Use nag_library, Only: f04fff, nag_wp
!     .. Implicit None Statement ..
      Implicit None
!     .. Parameters ..
      Integer, Parameter                 :: nin = 5, nout = 6
!     .. Local Scalars ..
      Integer                            :: ifail, n
      Logical                            :: wantp
!     .. Local Arrays ..
      Real (Kind=nag_wp), Allocatable  :: b(:), p(:), t(:), work(:), x(:)
!     .. Executable Statements ..
      Write (nout,*) 'F04FFF Example Program Results'
!     Skip heading in data file
      Read (nin,*)
      Read (nin,*) n
      Write (nout,*)
      Allocate (b(n),p(n-1),t(0:n-1),work(2*(n-1)),x(n))
      Read (nin,*) t(0:n-1)
      Read (nin,*) b(1:n)
      wantp = .True.

!     ifail: behaviour on error exit
!             =0 for hard exit, =1 for quiet-soft, =-1 for noisy-soft
      ifail = 1
      Call f04fff(n,t,b,x,wantp,p,work,ifail)

      If (ifail==0) Then
        Write (nout,*)
        Write (nout,*) 'Solution vector'
        Write (nout,99998) x(1:n)
        If (wantp) Then
          Write (nout,*)
          Write (nout,*) 'Reflection coefficients'
          Write (nout,99998) p(1:n-1)
        End If
      Else If (ifail>0) Then
        Write (nout,*)
        Write (nout,99999) 'Solution for system of order', ifail - 1
        Write (nout,99998) x(1:ifail-1)
        If (wantp) Then
          Write (nout,*)
          Write (nout,*) 'Reflection coefficients'
          Write (nout,99998) p(1:ifail-1)
        End If
      Else
        Write (nout,99997) ifail
      End If

99999 Format (1X,A,I5)
99998 Format (1X,5F9.4)
99997 Format (1X,' ** F04FFF returned with IFAIL = ',I5)
      End Program f04fffe
```

9.2 Program Data

```
F04FFF Example Program Data

   4                 : n
   4.0   3.0   2.0   1.0 : vector T
   1.0   1.0   1.0   1.0 : vector B
```

9.3   Program Results

```
F04FFF Example Program Results


Solution vector
   0.2000   0.0000  -0.0000   0.2000

Reflection coefficients
  -0.7500   0.1429   0.1667
```

_____