NAG Library Function Document

nag_jacobian_theta (s21ccc)

1 Purpose

nag_jacobian_theta (s21ccc) returns the value of one of the Jacobian theta functions $\theta_0(x, q)$, $\theta_1(x, q)$, $\theta_2(x, q)$, $\theta_3(x, q)$ or $\theta_4(x, q)$ for a real argument x and non-negative $q < 1$.

2 Specification

```
#include <nag.h>
#include <nags.h>
```

```
double nag_jacobian_theta (Integer k, double x, double q, NagError *fail)
```

3 Description

nag_jacobian_theta (s21ccc) evaluates an approximation to the Jacobian theta functions $\theta_0(x, q)$, $\theta_1(x, q)$, $\theta_2(x, q)$, $\theta_3(x, q)$ and $\theta_4(x, q)$ given by

$$\begin{aligned}
\theta_0(x,q) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2n\pi x), \\
\theta_1(x,q) &= 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin\{(2n+1)\pi x\}, \\
\theta_2(x,q) &= 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos\{(2n+1)\pi x\}, \\
\theta_3(x,q) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2n\pi x), \\
\theta_4(x,q) &= \theta_0(x,q),
\end{aligned}$$

where x and q (the *nome*) are real with $0 \leq q < 1$.

These functions are important in practice because every one of the Jacobian elliptic functions (see nag_jacobian_elliptic (s21cbc)) can be expressed as the ratio of two Jacobian theta functions (see Whittaker and Watson (1990)). There is also a bewildering variety of notations used in the literature to define them. Some authors (e.g., Section 16.27 of Abramowitz and Stegun (1972)) define the argument in the trigonometric terms to be x instead of πx. This can often lead to confusion, so great care must therefore be exercised when consulting the literature. Further details (including various relations and identities) can be found in the references.

nag_jacobian_theta (s21ccc) is based on a truncated series approach. If t differs from x or $-x$ by an integer when $0 \leq t \leq \frac{1}{2}$, it follows from the periodicity and symmetry properties of the functions that $\theta_1(x, q) = \pm\theta_1(t, q)$ and $\theta_3(x, q) = \pm\theta_3(t, q)$. In a region for which the approximation is sufficiently accurate, θ_1 is set equal to the first term ($n = 0$) of the transformed series

$$\theta_1(t,q) = 2\sqrt{\frac{\lambda}{\pi}}e^{-\lambda t^2}\sum_{n=0}^{\infty}(-1)^n e^{-\lambda\left(n+\frac{1}{2}\right)^2}\sinh\{(2n+1)\lambda t\}$$

and θ_3 is set equal to the first two terms (i.e., $n \leq 1$) of

$$\theta_3(t,q) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda t^2} \left\{ 1 + 2\sum_{n=1}^{\infty} e^{-\lambda n^2} \cosh(2n\lambda t) \right\},$$

where $\lambda = \pi^2/|\log_e q|$. Otherwise, the trigonometric series for $\theta_1(t,q)$ and $\theta_3(t,q)$ are used. For all values of x, θ_0 and θ_2 are computed from the relations $\theta_0(x,q) = \theta_3\left(\frac{1}{2} - |x|, q\right)$ and $\theta_2(x,q) = \theta_1\left(\frac{1}{2} - |x|, q\right)$.

4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

Byrd P F and Friedman M D (1971) *Handbook of Elliptic Integrals for Engineers and Scientists* pp. 315–320 (2nd Edition) Springer–Verlag

Magnus W, Oberhettinger F and Soni R P (1966) *Formulas and Theorems for the Special Functions of Mathematical Physics* 371–377 Springer–Verlag

Tølke F (1966) *Praktische Funktionenlehre (Bd. II)* 1–38 Springer–Verlag

Whittaker E T and Watson G N (1990) *A Course in Modern Analysis* (4th Edition) Cambridge University Press

5 Arguments

1: **k** – Integer *Input*

On entry: denotes the function $\theta_k(x,q)$ to be evaluated. Note that $\mathbf{k} = 4$ is equivalent to $\mathbf{k} = 0$.

Constraint: $0 \leq \mathbf{k} \leq 4$.

2: **x** – double *Input*

On entry: the argument x of the function.

3: **q** – double *Input*

On entry: the argument q of the function.

Constraint: $0.0 \leq \mathbf{q} < 1.0$.

4: **fail** – NagError * *Input/Output*

The NAG error argument (see Section 3.6 in the Essential Introduction).

6 Error Indicators and Warnings

NE_INT

On entry, $\mathbf{k} = \langle value \rangle$.
Constraint: $\mathbf{k} \leq 4$.

On entry, $\mathbf{k} = \langle value \rangle$.
Constraint: $\mathbf{k} \geq 0$.

NE_INTERNAL_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

NE_REAL

On entry, $\mathbf{q} = \langle value \rangle$.
Constraint: $\mathbf{q} < 1.0$.

On entry, $\mathbf{q} = \langle value \rangle$.
Constraint: $\mathbf{q} \geq 0.0$.

7 Accuracy

In principle the function is capable of achieving full relative precision in the computed values. However, the accuracy obtainable in practice depends on the accuracy of the standard elementary functions such as sin and cos.

8 Parallelism and Performance

Not applicable.

9 Further Comments

None.

10 Example

This example evaluates $\theta_2(x,q)$ at $x = 0.7$ when $q = 0.4$, and prints the results.

10.1 Program Text

```
/* nag_jacobian_theta (s21ccc) Example Program.
 *
 * Copyright 2000 Numerical Algorithms Group.
 *
 * NAG C Library
 *
 * Mark 6, 2000.
 */

#include <stdio.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nags.h>

int main(void)
{
  Integer  exit_status = 0, k;
  NagError fail;
  double   q, x, y;

  INIT_FAIL(fail);

  /* Skip heading in data file */
  scanf("%*[^\n] ");
  printf("nag_jacobian_theta (s21ccc) Example Program Results\n");
  printf(" k    x     q         y\n");
  while (scanf("%ld%lf%lf%*[^\n]", &k, &x, &q) != EOF)
    {
      /* nag_jacobian_theta (s21ccc).
       * Jacobian theta functions with real arguments
       */
      y = nag_jacobian_theta(k, x, q, &fail);
      if (fail.code != NE_NOERROR)
        {
          printf("Error from nag_jacobian_theta (s21ccc).\n%s\n",
                 fail.message);
          exit_status = 1;
```

```
          goto END;
        }
      printf("%2ld  %4.1f  %4.1f %13.4e\n", k, x, q, y);
    }
 END:
  return exit_status;
}
```

10.2 Program Data

```
nag_jacobian_theta (s21ccc) Example Program Data
 2  0.7  0.4  : Values of k, x and q
```

10.3 Program Results

```
nag_jacobian_theta (s21ccc) Example Program Results
 k    x     q         y
 2   0.7   0.4  -6.9289e-01
```