

## NAG Library Function Document

### nag\_sparse\_nherm\_fac (f11dnc)

#### 1 Purpose

nag\_sparse\_nherm\_fac (f11dnc) computes an incomplete  $LU$  factorization of a complex sparse non-Hermitian matrix, represented in coordinate storage format. This factorization may be used as a preconditioner in combination with nag\_sparse\_nherm\_basic\_solver (f11bsc) or nag\_sparse\_nherm\_fac\_sol (f11dq).

#### 2 Specification

```
#include <nag.h>
#include <nagf11.h>

void nag_sparse_nherm_fac (Integer n, Integer nnz, Complex a[], Integer la,
    Integer irow[], Integer icol[], Integer lfill, double dtol,
    Nag_SparseNsym_Piv pstrat, Nag_SparseNsym_Fact milu, Integer ipivp[],
    Integer ipivq[], Integer istr[], Integer idiag[], Integer *nnzc,
    Integer *npivm, NagError *fail)
```

#### 3 Description

nag\_sparse\_nherm\_fac (f11dnc) computes an incomplete  $LU$  factorization (see Meijerink and Van der Vorst (1977) and Meijerink and Van der Vorst (1981)) of a complex sparse non-Hermitian  $n$  by  $n$  matrix  $A$ . The factorization is intended primarily for use as a preconditioner with one of the iterative solvers nag\_sparse\_nherm\_basic\_solver (f11bsc) or nag\_sparse\_nherm\_fac\_sol (f11dq).

The decomposition is written in the form

$$A = M + R,$$

where

$$M = PLDUQ$$

and  $L$  is lower triangular with unit diagonal elements,  $D$  is diagonal,  $U$  is upper triangular with unit diagonals,  $P$  and  $Q$  are permutation matrices, and  $R$  is a remainder matrix.

The amount of fill-in occurring in the factorization can vary from zero to complete fill, and can be controlled by specifying either the maximum level of fill **lfill**, or the drop tolerance **dtol**.

The argument **pstrat** defines the pivoting strategy to be used. The options currently available are no pivoting, user-defined pivoting, partial pivoting by columns for stability, and complete pivoting by rows for sparsity and by columns for stability. The factorization may optionally be modified to preserve the row-sums of the original matrix.

The sparse matrix  $A$  is represented in coordinate storage (CS) format (see Section 2.1.1 in the f11 Chapter Introduction). The array **a** stores all the nonzero elements of the matrix  $A$ , while arrays **irow** and **icol** store the corresponding row and column indices respectively. Multiple nonzero elements may not be specified for the same row and column index.

The preconditioning matrix  $M$  is returned in terms of the CS representation of the matrix

$$C = L + D^{-1} + U - 2I.$$

Further algorithmic details are given in Section 9.3.

## 4 References

Meijerink J and Van der Vorst H (1977) An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix *Math. Comput.* **31** 148–162

Meijerink J and Van der Vorst H (1981) Guidelines for the usage of incomplete decompositions in solving sets of linear equations as they occur in practical problems *J. Comput. Phys.* **44** 134–155

## 5 Arguments

- 1: **n** – Integer *Input*  
*On entry:*  $n$ , the order of the matrix  $A$ .  
*Constraint:*  $n \geq 1$ .
- 2: **nnz** – Integer *Input*  
*On entry:* the number of nonzero elements in the matrix  $A$ .  
*Constraint:*  $1 \leq \mathbf{nnz} \leq n^2$ .
- 3: **a[la]** – Complex *Input/Output*  
*On entry:* the nonzero elements in the matrix  $A$ , ordered by increasing row index, and by increasing column index within each row. Multiple entries for the same row and column indices are not permitted. The function `nag_sparse_nherm_sort (f11znc)` may be used to order the elements in this way.  
*On exit:* the first **nnz** entries of **a** contain the nonzero elements of  $A$  and the next **nnzc** entries contain the elements of the matrix  $C$ . Matrix elements are ordered by increasing row index, and by increasing column index within each row.
- 4: **la** – Integer *Input*  
*On entry:* the dimension of the arrays **a**, **irow** and **icol**. These arrays must be of sufficient size to store both  $A$  (**nnz** elements) and  $C$  (**nnzc** elements).  
*Constraint:*  $la \geq 2 \times \mathbf{nnz}$ .
- 5: **irow[la]** – Integer *Input/Output*  
6: **icol[la]** – Integer *Input/Output*  
*On entry:* the row and column indices of the nonzero elements supplied in **a**.  
*Constraints:*  
**irow** and **icol** must satisfy these constraints (which may be imposed by a call to `nag_sparse_nherm_sort (f11znc)`):  
 $1 \leq \mathbf{irow}[i-1] \leq n$  and  $1 \leq \mathbf{icol}[i-1] \leq n$ , for  $i = 1, 2, \dots, \mathbf{nnz}$ ;  
either  $\mathbf{irow}[i-1] < \mathbf{irow}[i]$  or both  $\mathbf{irow}[i-1] = \mathbf{irow}[i]$  and  $\mathbf{icol}[i-1] < \mathbf{icol}[i]$ , for  $i = 1, 2, \dots, \mathbf{nnz}$ .  
*On exit:* the row and column indices of the nonzero elements returned in **a**.
- 7: **lfill** – Integer *Input*  
*On entry:* if **lfill**  $\geq 0$  its value is the maximum level of fill allowed in the decomposition (see Section 9.2). A negative value of **lfill** indicates that **dtol** will be used to control the fill instead.

- 8: **dtol** – double *Input*  
*On entry:* if **lfill** < 0, **dtol** is used as a drop tolerance to control the fill-in (see Section 9.2); otherwise **dtol** is not referenced.  
*Constraint:* if **lfill** < 0, **dtol** ≥ 0.0.
- 9: **pstrat** – Nag\_SparseNsym\_Piv *Input*  
*On entry:* specifies the pivoting strategy to be adopted.  
**pstrat** = Nag\_SparseNsym\_NoPiv  
 No pivoting is carried out.  
**pstrat** = Nag\_SparseNsym\_UserPiv  
 Pivoting is carried out according to the user-defined input values of **ipivp** and **ipivq**.  
**pstrat** = Nag\_SparseNsym\_PartialPiv  
 Partial pivoting by columns for stability is carried out.  
**pstrat** = Nag\_SparseNsym\_CompletePiv  
 Complete pivoting by rows for sparsity, and by columns for stability, is carried out.  
*Suggested value:* **pstrat** = Nag\_SparseNsym\_CompletePiv.  
*Constraint:* **pstrat** = Nag\_SparseNsym\_NoPiv, Nag\_SparseNsym\_UserPiv, Nag\_SparseNsym\_PartialPiv or Nag\_SparseNsym\_CompletePiv.
- 10: **milu** – Nag\_SparseNsym\_Fact *Input*  
*On entry:* indicates whether or not the factorization should be modified to preserve row-sums (see Section 9.4).  
**milu** = Nag\_SparseNsym\_ModFact  
 The factorization is modified.  
**milu** = Nag\_SparseNsym\_UnModFact  
 The factorization is not modified.  
*Constraint:* **milu** = Nag\_SparseNsym\_ModFact or Nag\_SparseNsym\_UnModFact.
- 11: **ipivp**[**n**] – Integer *Input/Output*
- 12: **ipivq**[**n**] – Integer *Input/Output*  
*On entry:* if **pstrat** = Nag\_SparseNsym\_UserPiv, then **ipivp**[ $k - 1$ ] and **ipivq**[ $k - 1$ ] must specify the row and column indices of the element used as a pivot at elimination stage  $k$ . Otherwise **ipivp** and **ipivq** need not be initialized.  
*Constraint:* if **pstrat** = Nag\_SparseNsym\_UserPiv, **ipivp** and **ipivq** must both hold valid permutations of the integers on [1,**n**].  
*On exit:* the pivot indices. If **ipivp**[ $k - 1$ ] =  $i$  and **ipivq**[ $k - 1$ ] =  $j$  then the element in row  $i$  and column  $j$  was used as the pivot at elimination stage  $k$ .
- 13: **istr**[**n** + 1] – Integer *Output*  
*On exit:* **istr**[ $i - 1$ ] – 1, for  $i = 1, 2, \dots, \mathbf{n}$ , is the starting address in the arrays **a**, **row** and **icol** of row  $i$  of the matrix  $C$ . **istr**[**n**] – 1 is the address of the last nonzero element in  $C$  plus one.
- 14: **idiag**[**n**] – Integer *Output*  
*On exit:* **idiag**[ $i - 1$ ], for  $i = 1, 2, \dots, \mathbf{n}$ , holds the index of arrays **a**, **row** and **icol** which holds the diagonal element in row  $i$  of the matrix  $C$ .
- 15: **nnzc** – Integer \* *Output*  
*On exit:* the number of nonzero elements in the matrix  $C$ .

- 16: **npivm** – Integer \* *Output*  
*On exit:* if **npivm** > 0 it gives the number of pivots which were modified during the factorization to ensure that  $M$  exists.  
 If **npivm** = -1 no pivot modifications were required, but a local restart occurred (see Section 9.3). The quality of the preconditioner will generally depend on the returned value of **npivm**.  
 If **npivm** is large the preconditioner may not be satisfactory. In this case it may be advantageous to call `nag_sparse_nherm_fac (f11dnc)` again with an increased value of **lfill**, a reduced value of **dtol**, or set **pstrat** = Nag\_SparseNsym\_CompletePiv. See also Section 9.5.
- 17: **fail** – NagError \* *Input/Output*  
 The NAG error argument (see Section 3.6 in the Essential Introduction).

## 6 Error Indicators and Warnings

### NE\_ALLOC\_FAIL

Dynamic memory allocation failed.

### NE\_BAD\_PARAM

On entry, argument  $\langle value \rangle$  had an illegal value.

### NE\_INT

On entry, **n** =  $\langle value \rangle$ .

Constraint: **n**  $\geq 1$ .

On entry, **nnz** =  $\langle value \rangle$ .

Constraint: **nnz**  $\geq 1$ .

### NE\_INT\_2

On entry, **la** =  $\langle value \rangle$  and **nnz** =  $\langle value \rangle$ .

Constraint: **la**  $\geq 2 \times \mathbf{nnz}$

On entry, **nnz** =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .

Constraint: **nnz**  $\leq \mathbf{n}^2$

### NE\_INTERNAL\_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

### NE\_INVALID\_CS

On entry,  $I = \langle value \rangle$ , **icol**[ $I - 1$ ] =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .

Constraint:  $1 \leq \mathbf{icol}[i] \leq \mathbf{n}$ , for  $i = 0, 1, \dots, \mathbf{nnz} - 1$

On entry,  $I = \langle value \rangle$ , **irow**[ $I - 1$ ] =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .

Constraint:  $1 \leq \mathbf{irow}[i - 1] \leq \mathbf{n}$ , for  $i = 1, 2, \dots, \mathbf{nnz}$

### NE\_INVALID\_ROWCOL\_PIVOT

On entry, a user-supplied value of **ipivp** is repeated.

On entry, a user-supplied value of **ipivp** lies outside the range  $[1, \mathbf{n}]$ .

On entry, a user-supplied value of **ipivq** is repeated.

On entry, a user-supplied value of **ipivq** lies outside the range  $[1, \mathbf{n}]$ .

**NE\_NOT\_STRICTLY\_INCREASING**

On entry,  $\mathbf{a}[i - 1]$  is out of order:  $i = \langle \text{value} \rangle$ .

On entry, the location ( $\mathbf{irow}[I - 1], \mathbf{icol}[I - 1]$ ) is a duplicate:  $I = \langle \text{value} \rangle$ .

**NE\_REAL**

On entry,  $\mathbf{dtol} = \langle \text{value} \rangle$ .

Constraint:  $\mathbf{dtol} \geq 0.0$

**NE\_TOO\_SMALL**

The number of nonzero entries in the decomposition is too large. The decomposition has been terminated before completion. Either increase  $\mathbf{la}$ , or reduce the fill by reducing  $\mathbf{lfill}$ , or increasing  $\mathbf{dtol}$ .

**7 Accuracy**

The accuracy of the factorization will be determined by the size of the elements that are dropped and the size of any modifications made to the pivot elements. If these sizes are small then the computed factors will correspond to a matrix close to  $A$ . The factorization can generally be made more accurate by increasing  $\mathbf{lfill}$ , or by reducing  $\mathbf{dtol}$  with  $\mathbf{lfill} < 0$ .

If `nag_sparse_nherm_fac` (f11dnc) is used in combination with `nag_sparse_nherm_basic_solver` (f11bse) or `nag_sparse_nherm_fac_sol` (f11dq), the more accurate the factorization the fewer iterations will be required. However, the cost of the decomposition will also generally increase.

**8 Parallelism and Performance**

Not applicable.

**9 Further Comments****9.1 Timing**

The time taken for a call to `nag_sparse_nherm_fac` (f11dnc) is roughly proportional to  $\mathbf{nnzc}^2/\mathbf{n}$ .

**9.2 Control of Fill-in**

If  $\mathbf{lfill} \geq 0$  the amount of fill-in occurring in the incomplete factorization is controlled by limiting the maximum level of fill-in to  $\mathbf{lfill}$ . The original nonzero elements of  $A$  are defined to be of level 0. The fill level of a new nonzero location occurring during the factorization is defined as:

$$k = \max(k_e, k_c) + 1,$$

where  $k_e$  is the level of fill of the element being eliminated, and  $k_c$  is the level of fill of the element causing the fill-in.

If  $\mathbf{lfill} < 0$  the fill-in is controlled by means of the drop tolerance  $\mathbf{dtol}$ . A potential fill-in element  $a_{ij}$  occurring in row  $i$  and column  $j$  will not be included if:

$$|a_{ij}| < \mathbf{dtol} \times \alpha,$$

where  $\alpha$  is the maximum modulus element in the matrix  $A$ .

For either method of control, any elements which are not included are discarded unless  $\mathbf{milu} = \text{Nag\_SparseNsym\_ModFact}$ , in which case their contributions are subtracted from the pivot element in the relevant elimination row, to preserve the row-sums of the original matrix.

Should the factorization process break down a local restart process is implemented as described in Section 9.3. This will affect the amount of fill present in the final factorization.

### 9.3 Algorithmic Details

The factorization is constructed row by row. At each elimination stage a row index is chosen. In the case of complete pivoting this index is chosen in order to reduce fill-in. Otherwise the rows are treated in the order given, or some user-defined order.

The chosen row is copied from the original matrix  $A$  and modified according to those previous elimination stages which affect it. During this process any fill-in elements are either dropped or kept according to the values of **lfill** or **dtol**. In the case of a modified factorization (**milu** = Nag\_SparseNsym\_ModFact) the sum of the dropped terms for the given row is stored.

Finally the pivot element for the row is chosen and the multipliers are computed for this elimination stage. For partial or complete pivoting the pivot element is chosen in the interests of stability as the element of largest absolute value in the row. Otherwise the pivot element is chosen in the order given, or some user-defined order.

If the factorization breaks down because the chosen pivot element is zero, or there is no nonzero pivot available, a local restart recovery process is implemented. The modification of the given pivot row according to previous elimination stages is repeated, but this time keeping all fill-in. Note that in this case the final factorization will include more fill than originally specified by the user-supplied value of **lfill** or **dtol**. The local restart usually results in a suitable nonzero pivot arising. The original criteria for dropping fill-in elements is then resumed for the next elimination stage (hence the **local** nature of the restart process). Should this restart process also fail to produce a nonzero pivot element an arbitrary unit pivot is introduced in an arbitrarily chosen column. `nag_sparse_nherm_fac` (f11dnc) returns an integer argument **npivm** which gives the number of these arbitrary unit pivots introduced. If no pivots were modified but local restarts occurred **npivm** = -1 is returned.

### 9.4 Choice of Arguments

There is unfortunately no choice of the various algorithmic arguments which is optimal for all types of matrix, and some experimentation will generally be required for each new type of matrix encountered. The recommended approach is to start with **lfill** = 0 and **pstrat** = Nag\_SparseNsym\_CompletePiv. If the value returned for **npivm** is significantly larger than zero, i.e., a large number of pivot modifications were required to ensure that  $M$  existed, the preconditioner is not likely to be satisfactory. In this case increase **lfill** until **npivm** falls to a value close to zero.

For certain classes of matrices (typically those arising from the discretization of elliptic or parabolic partial differential equations) the convergence rate of the preconditioned iterative solver can sometimes be significantly improved by using an incomplete factorization which preserves the row-sums of the original matrix. In these cases try setting **milu** = Nag\_SparseNsym\_ModFact.

### 9.5 Direct Solution of Sparse Linear Systems

Although it is not the primary purpose of the functions `nag_sparse_nherm_fac` (f11dnc) and `nag_sparse_nherm_precon_ilu_solve` (f11dpc), they may be used together to obtain a **direct** solution to a nonsingular sparse complex non-Hermitian linear system. To achieve this the call to `nag_sparse_nherm_precon_ilu_solve` (f11dpc) should be preceded by a **complete**  $LU$  factorization

$$A = PLDUQ = M.$$

A complete factorization is obtained from a call to `nag_sparse_nherm_fac` (f11dnc) with **lfill** < 0 and **dtol** = 0.0, provided **npivm** ≤ 0 on exit. A positive value of **npivm** indicates that  $A$  is singular, or ill-conditioned. A factorization with positive **npivm** may serve as a preconditioner, but will not result in a direct solution. It is therefore **essential** to check the output value of **npivm** if a direct solution is required.

The use of `nag_sparse_nherm_fac` (f11dnc) and `nag_sparse_nherm_precon_ilu_solve` (f11dpc) as a direct method is illustrated in `nag_sparse_nherm_precon_ilu_solve` (f11dpc).

## 10 Example

This example reads in a complex sparse non-Hermitian matrix  $A$  and calls `nag_sparse_nherm_fac` (`f11dnc`) to compute an incomplete  $LU$  factorization. It then outputs the nonzero elements of both  $A$  and  $C = L + D^{-1} + U - 2I$ .

The call to `nag_sparse_nherm_fac` (`f11dnc`) has `lfill = 0`, and `pstrat = Nag_SparseNsym_CompletePiv`, giving an unmodified zero-fill  $LU$  factorization, with row pivoting for sparsity and column pivoting for stability.

### 10.1 Program Text

```

/* nag_sparse_nherm_fac (f11dnc) Example Program.
 *
 * Copyright 2011, Numerical Algorithms Group.
 *
 * Mark 23, 2011.
 */
#include <nag.h>
#include <nag_stdlib.h>
#include <naga02.h>
#include <nagf11.h>
int main(void)
{
    /* Scalars */
    Integer          exit_status = 0;
    double           dtol;
    Integer          i, la, lfill, n, nnz, nnzc, npivm;
    /* Arrays */
    Complex          *a = 0;
    Integer          *icol = 0, *idiag = 0, *ipivp = 0, *ipivq = 0,
                    *irow = 0, *istr = 0;
    char             nag_enum_arg[40];
    /* NAG types */
    Nag_SparseNsym_Piv pstrat;
    Nag_SparseNsym_Fact milu;
    NagError         fail;

    INIT_FAIL(fail);

    printf("nag_sparse_nherm_fac (f11dnc) Example Program Results \n\n");
    /* Skip heading in data file*/
    scanf("%s[^\n] ");
    /* Read algorithmic parameters*/
    scanf("%ld%*[^\n]", &n);
    scanf("%ld%*[^\n]", &nnz);
    scanf("%ld%lf%*[^\n]", &llfill, &dtol);
    scanf("%39s%*[^\n]", nag_enum_arg);
    /* nag_enum_name_to_value (x04nac).
     * Converts NAG enum member name to value
     */
    pstrat = (Nag_SparseNsym_Piv) nag_enum_name_to_value(nag_enum_arg);
    scanf("%39s%*[^\n]", nag_enum_arg);
    milu = (Nag_SparseNsym_Fact) nag_enum_name_to_value(nag_enum_arg);
    la = 2 * nnz;
    if (
        !(a = NAG_ALLOC((la), Complex)) ||
        !(icol = NAG_ALLOC((la), Integer)) ||
        !(idiag = NAG_ALLOC((n), Integer)) ||
        !(ipivp = NAG_ALLOC((n), Integer)) ||
        !(ipivq = NAG_ALLOC((n), Integer)) ||
        !(irow = NAG_ALLOC((la), Integer)) ||
        !(istr = NAG_ALLOC((n + 1), Integer))
    ) {
        printf("Allocation failure\n");
        exit_status = -1;
        goto END;
    }
}

```

```

/* Read the matrix a */
for (i = 0; i < nnz; ++i)
    scanf(" ( %lf , %lf ) %ld%ld%*[\n]",
          &a[i].re, &a[i].im, &irow[i], &icol[i]);

/* Calculate incomplete LU factorization */
/* nag_sparse_nherm_fac (f11dnc)
 * Complex sparse non-Hermitian linear systems, incomplete LU factorization
 */
nag_sparse_nherm_fac(n, nnz, a, la, irow, icol, lfill, dtol, pstrat, milu,
                    ipivp, ipivq, istr, idiag, &nnzc, &npivm, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_sparse_nsym_fac (f11dac).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* Output original matrix*/
printf(" Original Matrix\n n      = %6ld\n", n);
printf(" nnz      = %6ld\n", nnz);

for (i = 0; i < nnz; ++i)
    printf("%8ld (%16.4e, %16.4e) %8ld%8ld\n",
          i, a[i].re, a[i].im, irow[i], icol[i]);
printf("\n");

/* Output details of the factorization*/
printf(" Factorization\n n      = %6ld\n", n);
printf(" nnz      = %6ld\n", nnzc);
printf(" npivm = %6ld\n", npivm);

for (i = nnz; i < nnz + nnzc; ++i)
    printf("%8ld (%16.4e, %16.4e) %8ld%8ld\n",
          i, a[i].re, a[i].im, irow[i], icol[i]);

printf("\n      i      ipivp[i] ipivq[i]\n"); /* */
for (i = 0; i < n; ++i)
    printf("%10ld%10ld%10ld\n", i, ipivp[i], ipivq[i]);
END:
NAG_FREE(a);
NAG_FREE(icol);
NAG_FREE(idiag);
NAG_FREE(ipivp);
NAG_FREE(ipivq);
NAG_FREE(irow);
NAG_FREE(istr);
return exit_status;
}

```

## 10.2 Program Data

nag\_sparse\_nherm\_fac (f11dnc) Example Program Data

```

4          : n
11         : nnz
0          0.0          : lfill, dtol
Nag_SparseNsym_CompletePiv : pstrat
Nag_SparseNsym_UnModFact   : milu
( 1., 3.)  1  2
( 1., 0.)  1  3
(-1.,-2.)  2  1
( 2.,-2.)  2  3
( 2., 1.)  2  4
( 0., 5.)  3  1
(-2., 0.)  3  4
( 1., 1.)  4  1
(-2., 4.)  4  2
( 1.,-3.)  4  3
( 0., 7.)  4  4          : a[i], irow[i], icol[i], i=0,...,nnz-1

```



**10.3 Program Results**

nag\_sparse\_nherm\_fac (f11dnc) Example Program Results

Original Matrix

```

n      =      4
nnz    =     11
0 (    1.0000e+00,      3.0000e+00)    1    2
1 (    1.0000e+00,      0.0000e+00)    1    3
2 (   -1.0000e+00,     -2.0000e+00)    2    1
3 (    2.0000e+00,     -2.0000e+00)    2    3
4 (    2.0000e+00,      1.0000e+00)    2    4
5 (    0.0000e+00,      5.0000e+00)    3    1
6 (   -2.0000e+00,      0.0000e+00)    3    4
7 (    1.0000e+00,      1.0000e+00)    4    1
8 (   -2.0000e+00,      4.0000e+00)    4    2
9 (    1.0000e+00,     -3.0000e+00)    4    3
10 (    0.0000e+00,      7.0000e+00)    4    4

```

Factorization

```

n      =      4
nnz    =     11
npivm =      0
11 (    1.0000e-01,     -3.0000e-01)    1    1
12 (    1.0000e-01,     -3.0000e-01)    1    3
13 (    0.0000e+00,     -2.0000e-01)    2    2
14 (    0.0000e+00,      4.0000e-01)    2    4
15 (   -4.0000e-01,      2.0000e-01)    3    2
16 (    2.5000e-01,      2.5000e-01)    3    3
17 (   -5.0000e-02,      6.5000e-01)    3    4
18 (    1.0000e+00,      1.0000e+00)    4    1
19 (    2.0000e-01,     -2.0000e-01)    4    2
20 (    1.0000e+00,     -1.0000e+00)    4    3
21 (   -4.8035e-02,     -1.3974e-01)    4    4

```

```

i      ipivp[i]  ipivq[i]
0      1         2
1      3         1
2      2         3
3      4         4

```

---