# NAG Library Function Document

# nag_zgtrfs (f07cvc)

## 1 Purpose

nag_zgtrfs (f07cvc) computes error bounds and refines the solution to a complex system of linear equations $AX = B$ or $A^{\mathrm{T}}X = B$ or $A^{\mathrm{H}}X = B$, where $A$ is an $n$ by $n$ tridiagonal matrix and $X$ and $B$ are $n$ by $r$ matrices, using the $LU$ factorization returned by nag_zgttrf (f07crc) and an initial solution returned by nag_zgttrs (f07csc). Iterative refinement is used to reduce the backward error as much as possible.

## 2 Specification

```
#include <nag.h>
#include <nagf07.h>
void nag_zgtrfs (Nag_OrderType order, Nag_TransType trans, Integer n,
     Integer nrhs, const Complex dl[], const Complex d[], const Complex du[],
     const Complex dlf[], const Complex df[], const Complex duf[],
     const Complex du2[], const Integer ipiv[], const Complex b[],
     Integer pdb, Complex x[], Integer pdx, double ferr[], double berr[],
     NagError *fail)
```

## 3 Description

nag_zgtrfs (f07cvc) should normally be preceded by calls to nag_zgttrf (f07crc) and nag_zgttrs (f07csc). nag_zgttrf (f07crc) uses Gaussian elimination with partial pivoting and row interchanges to factorize the matrix $A$ as

$$A = PLU,$$

where $P$ is a permutation matrix, $L$ is unit lower triangular with at most one nonzero subdiagonal element in each column, and $U$ is an upper triangular band matrix, with two superdiagonals. nag_zgttrs (f07csc) then utilizes the factorization to compute a solution, $\hat{X}$, to the required equations. Letting $\hat{x}$ denote a column of $\hat{X}$, nag_zgtrfs (f07cvc) computes a *component-wise backward error*, $\beta$, the smallest relative perturbation in each element of $A$ and $b$ such that $\hat{x}$ is the exact solution of a perturbed system

$$(A + E)\hat{x} = b + f, \quad \text{with} \quad \left|e_{ij}\right| \leq \beta\left|a_{ij}\right|, \quad \text{and} \quad \left|f_j\right| \leq \beta\left|b_j\right|.$$

The function also estimates a bound for the *component-wise forward error* in the computed solution defined by $\max|x_i - \hat{x}_i|/\max|\hat{x}_i|$, where $x$ is the corresponding column of the exact solution, $X$.

## 4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia http://www.netlib.org/lapack/lug

## 5 Arguments

1: **order** – Nag_OrderType *Input*

*On entry*: the **order** argument specifies the two-dimensional storage scheme being used, i.e., row-major ordering or column-major ordering. C language defined storage is specified by **order** = Nag_RowMajor. See Section 3.2.1.3 in the Essential Introduction for a more detailed explanation of the use of this argument.

*Constraint*: **order** = Nag_RowMajor or Nag_ColMajor.

2: **trans** – Nag_TransType *Input*

*On entry*: specifies the equations to be solved as follows:

**trans** = Nag_NoTrans
Solve $AX = B$ for $X$.

**trans** = Nag_Trans
Solve $A^{\mathrm{T}}X = B$ for $X$.

**trans** = Nag_ConjTrans
Solve $A^{\mathrm{H}}X = B$ for $X$.

*Constraint*: **trans** = Nag_NoTrans, Nag_Trans or Nag_ConjTrans.

3: **n** – Integer *Input*

*On entry*: $n$, the order of the matrix $A$.

*Constraint*: $\mathbf{n} \geq 0$.

4: **nrhs** – Integer *Input*

*On entry*: $r$, the number of right-hand sides, i.e., the number of columns of the matrix $B$.

*Constraint*: $\mathbf{nrhs} \geq 0$.

5: **dl**[$dim$] – const Complex *Input*

**Note**: the dimension, *dim*, of the array **dl** must be at least $\max(1, \mathbf{n} - 1)$.

*On entry*: must contain the $(n - 1)$ subdiagonal elements of the matrix $A$.

6: **d**[$dim$] – const Complex *Input*

**Note**: the dimension, *dim*, of the array **d** must be at least $\max(1, \mathbf{n})$.

*On entry*: must contain the $n$ diagonal elements of the matrix $A$.

7: **du**[$dim$] – const Complex *Input*

**Note**: the dimension, *dim*, of the array **du** must be at least $\max(1, \mathbf{n} - 1)$.

*On entry*: must contain the $(n - 1)$ superdiagonal elements of the matrix $A$.

8: **dlf**[$dim$] – const Complex *Input*

**Note**: the dimension, *dim*, of the array **dlf** must be at least $\max(1, \mathbf{n} - 1)$.

*On entry*: must contain the $(n - 1)$ multipliers that define the matrix $L$ of the $LU$ factorization of $A$.

9: **df**[$dim$] – const Complex *Input*

**Note**: the dimension, *dim*, of the array **df** must be at least $\max(1, \mathbf{n})$.

*On entry*: must contain the $n$ diagonal elements of the upper triangular matrix $U$ from the $LU$ factorization of $A$.

10: **duf**[$dim$] – const Complex *Input*

**Note**: the dimension, *dim*, of the array **duf** must be at least $\max(1, \mathbf{n} - 1)$.

*On entry*: must contain the $(n - 1)$ elements of the first superdiagonal of $U$.

11: **du2**[$dim$] – const Complex *Input*

**Note**: the dimension, *dim*, of the array **du2** must be at least $\max(1, \mathbf{n} - 2)$.

*On entry*: must contain the $(n-2)$ elements of the second superdiagonal of $U$.

12: **ipiv**[$dim$] – const Integer *Input*

**Note**: the dimension, *dim*, of the array **ipiv** must be at least $\max(1, \mathbf{n})$.

*On entry*: must contain the $n$ pivot indices that define the permutation matrix $P$. At the $i$th step, row $i$ of the matrix was interchanged with row **ipiv**$[i-1]$, and **ipiv**$[i-1]$ must always be either $i$ or $(i+1)$, **ipiv**$[i-1] = i$ indicating that a row interchange was not performed.

13: **b**[$dim$] – const Complex *Input*

**Note**: the dimension, *dim*, of the array **b** must be at least

$\max(1, \mathbf{pdb} \times \mathbf{nrhs})$ when **order** = Nag_ColMajor;
$\max(1, \mathbf{n} \times \mathbf{pdb})$ when **order** = Nag_RowMajor.

The $(i,j)$th element of the matrix $B$ is stored in

**b**$[(j-1) \times \mathbf{pdb} + i - 1]$ when **order** = Nag_ColMajor;
**b**$[(i-1) \times \mathbf{pdb} + j - 1]$ when **order** = Nag_RowMajor.

*On entry*: the $n$ by $r$ matrix of right-hand sides $B$.

14: **pdb** – Integer *Input*

*On entry*: the stride separating row or column elements (depending on the value of **order**) in the array **b**.

*Constraints*:

if **order** = Nag_ColMajor, **pdb** $\geq \max(1, \mathbf{n})$;
if **order** = Nag_RowMajor, **pdb** $\geq \max(1, \mathbf{nrhs})$.

15: **x**[$dim$] – Complex *Input/Output*

**Note**: the dimension, *dim*, of the array **x** must be at least

$\max(1, \mathbf{pdx} \times \mathbf{nrhs})$ when **order** = Nag_ColMajor;
$\max(1, \mathbf{n} \times \mathbf{pdx})$ when **order** = Nag_RowMajor.

The $(i,j)$th element of the matrix $X$ is stored in

**x**$[(j-1) \times \mathbf{pdx} + i - 1]$ when **order** = Nag_ColMajor;
**x**$[(i-1) \times \mathbf{pdx} + j - 1]$ when **order** = Nag_RowMajor.

*On entry*: the $n$ by $r$ initial solution matrix $X$.

*On exit*: the $n$ by $r$ refined solution matrix $X$.

16: **pdx** – Integer *Input*

*On entry*: the stride separating row or column elements (depending on the value of **order**) in the array **x**.

*Constraints*:

if **order** = Nag_ColMajor, **pdx** $\geq \max(1, \mathbf{n})$;
if **order** = Nag_RowMajor, **pdx** $\geq \max(1, \mathbf{nrhs})$.

17: **ferr**[**nrhs**] – double *Output*

*On exit*: estimate of the forward error bound for each computed solution vector, such that $\left\|\hat{x}_j - x_j\right\|_\infty / \left\|\hat{x}_j\right\|_\infty \leq$ **ferr**$[j-1]$, where $\hat{x}_j$ is the $j$th column of the computed solution returned in the array **x** and $x_j$ is the corresponding column of the exact solution $X$. The estimate is almost always a slight overestimate of the true error.

18: **berr**[**nrhs**] – double *Output*

*On exit*: estimate of the component-wise relative backward error of each computed solution vector $\hat{x}_j$ (i.e., the smallest relative change in any element of $A$ or $B$ that makes $\hat{x}_j$ an exact solution).

19: **fail** – NagError * *Input/Output*

The NAG error argument (see Section 3.6 in the Essential Introduction).

## 6 Error Indicators and Warnings

**NE_ALLOC_FAIL**

Dynamic memory allocation failed.

**NE_BAD_PARAM**

On entry, argument ⟨*value*⟩ had an illegal value.

**NE_INT**

On entry, **n** = ⟨*value*⟩.
Constraint: **n** $\geq 0$.

On entry, **nrhs** = ⟨*value*⟩.
Constraint: **nrhs** $\geq 0$.

On entry, **pdb** = ⟨*value*⟩.
Constraint: **pdb** $> 0$.

On entry, **pdx** = ⟨*value*⟩.
Constraint: **pdx** $> 0$.

**NE_INT_2**

On entry, **pdb** = ⟨*value*⟩ and **n** = ⟨*value*⟩.
Constraint: **pdb** $\geq \max(1, \mathbf{n})$.

On entry, **pdb** = ⟨*value*⟩ and **nrhs** = ⟨*value*⟩.
Constraint: **pdb** $\geq \max(1, \mathbf{nrhs})$.

On entry, **pdx** = ⟨*value*⟩ and **n** = ⟨*value*⟩.
Constraint: **pdx** $\geq \max(1, \mathbf{n})$.

On entry, **pdx** = ⟨*value*⟩ and **nrhs** = ⟨*value*⟩.
Constraint: **pdx** $\geq \max(1, \mathbf{nrhs})$.

**NE_INTERNAL_ERROR**

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

## 7 Accuracy

The computed solution for a single right-hand side, $\hat{x}$, satisfies an equation of the form

$$(A + E)\hat{x} = b,$$

where

$$\|E\|_\infty = O(\epsilon)\|A\|_\infty$$

and $\epsilon$ is the ***machine precision***. An approximate error bound for the computed solution is given by

$$\frac{\|\hat{x} - x\|_\infty}{\|x\|_\infty} \leq \kappa(A)\frac{\|E\|_\infty}{\|A\|_\infty},$$

where $\kappa(A) = \left\|A^{-1}\right\|_\infty \|A\|_\infty$, the condition number of $A$ with respect to the solution of the linear equations. See Section 4.4 of Anderson *et al.* (1999) for further details.

Function nag_zgtcon (f07cuc) can be used to estimate the condition number of $A$.

## 8 Parallelism and Performance

nag_zgtrfs (f07cvc) is threaded by NAG for parallel execution in multithreaded implementations of the NAG Library.

nag_zgtrfs (f07cvc) makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the Users' Note for your implementation for any additional implementation-specific information.

## 9 Further Comments

The total number of floating-point operations required to solve the equations $AX = B$ or $A^{\mathrm{T}}X = B$ or $A^{\mathrm{H}}X = B$ is proportional to $nr$. At most five steps of iterative refinement are performed, but usually only one or two steps are required.

The real analogue of this function is nag_dgtrfs (f07chc).

## 10 Example

This example solves the equations

$$AX = B,$$

where $A$ is the tridiagonal matrix

$$A = \begin{pmatrix} -1.3+1.3i & 2.0-1.0i & 0 & 0 & 0 \\ 1.0-2.0i & -1.3+1.3i & 2.0+1.0i & 0 & 0 \\ 0 & 1.0+1.0i & -1.3+3.3i & -1.0+1.0i & 0 \\ 0 & 0 & 2.0-3.0i & -0.3+4.3i & 1.0-1.0i \\ 0 & 0 & 0 & 1.0+1.0i & -3.3+1.3i \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2.4-5.0i & 2.7+6.9i \\ 3.4+18.2i & -6.9-5.3i \\ -14.7+9.7i & -6.0-0.6i \\ 31.9-7.7i & -3.9+9.3i \\ -1.0+1.6i & -3.0+12.2i \end{pmatrix}.$$

Estimates for the backward errors and forward errors are also output.

### 10.1 Program Text

```
/* nag_zgtrfs (f07cvc) Example Program.
 *
 * Copyright 2004 Numerical Algorithms Group.
 *
 * Mark 23, 2011
 */
#include <stdio.h>
#include <nag.h>
#include <nagx04.h>
#include <nag_stdlib.h>
#include <nagf07.h>
#include <nagf16.h>

int main(void)
```

```
{
  /* Scalars */
  Integer       exit_status = 0, i, j, n, nrhs, pdb, pdx;

  /* Arrays */
  Complex       *b = 0, *d = 0, *df = 0, *dl = 0, *dlf = 0, *du = 0,
                *du2 = 0, *duf = 0, *x = 0;
  double        *berr = 0, *ferr = 0;
  Integer       *ipiv = 0;

  /* Nag Types */
  NagError      fail;
  Nag_OrderType order;

#ifdef NAG_COLUMN_MAJOR
#define B(I, J) b[(J - 1) * pdb + I - 1]
  order = Nag_ColMajor;
#else
#define B(I, J) b[(I - 1) * pdb + J - 1]
  order = Nag_RowMajor;
#endif

  INIT_FAIL(fail);

  printf("nag_zgtrfs (f07cvc) Example Program Results\n\n");
  /* Skip heading in data file */
  scanf("%*[^\n]");
  scanf("%ld%ld%*[^\n]", &n, &nrhs);
  if (n < 0 || nrhs < 0)
    {
      printf("Invalid n or nrhs\n");
      exit_status = 1;
      goto END;
    }

  if (!(b    = NAG_ALLOC(n * nrhs, Complex)) ||
      !(d    = NAG_ALLOC(n, Complex)) ||
      !(df   = NAG_ALLOC(n, Complex)) ||
      !(dl   = NAG_ALLOC(n - 1, Complex)) ||
      !(dlf  = NAG_ALLOC(n - 1, Complex)) ||
      !(du   = NAG_ALLOC(n - 1, Complex)) ||
      !(du2  = NAG_ALLOC(n - 2, Complex)) ||
      !(duf  = NAG_ALLOC(n - 1, Complex)) ||
      !(x    = NAG_ALLOC(n * nrhs, Complex)) ||
      !(berr = NAG_ALLOC(nrhs, double)) ||
      !(ferr = NAG_ALLOC(nrhs, double)) ||
      !(ipiv = NAG_ALLOC(n, Integer)))
    {
      printf("Allocation failure\n");
      exit_status = -1;
      goto END;
    }
#ifdef NAG_COLUMN_MAJOR
  pdb = n;
  pdx = n;
#else
  pdb = nrhs;
  pdx = nrhs;
#endif

  /* Read the tridiagonal matrix A from data file */
  for (i = 0; i < n - 1; ++i) scanf(" ( %lf , %lf )", &du[i].re, &du[i].im);
  scanf("%*[^\n]");
  for (i = 0; i < n; ++i) scanf(" ( %lf , %lf )", &d[i].re, &d[i].im);
  scanf("%*[^\n]");
  for (i = 0; i < n - 1; ++i) scanf(" ( %lf , %lf )", &dl[i].re, &dl[i].im);
  scanf("%*[^\n]");

  /* Read the right hand matrix B */
  for (i = 1; i <= n; ++i)
    for (j = 1; j <= nrhs; ++j)
```

```
      scanf(" ( %lf , %lf )", &B(i, j).re, &B(i, j).im);
  scanf("%*[^\n]");


  /* Copy A into duf, df and dlf */
  for (i = 0; i < n - 1; ++i)
    {
      duf[i].re = du[i].re, duf[i].im = du[i].im;
      df[i].re = d[i].re, df[i].im = d[i].im;
      dlf[i].re = dl[i].re, dlf[i].im = dl[i].im;
    }
  df[n - 1].re = d[n - 1].re, df[n - 1].im = d[n - 1].im;

  /* copy B into X using nag_zge_copy (f16tfc). */
  nag_zge_copy(order, Nag_NoTrans, n, nrhs, b, pdb, x, pdx, &fail);
  if (fail.code != NE_NOERROR)
    {
      printf("Error from nag_zge_copy (f16tfc).\n%s\n", fail.message);
      exit_status = 1;
      goto END;
    }

  /* Factorize the copy of the tridiagonal matrix A using
   * nag_zgttrf (f07crc).
   */
  nag_zgttrf(n, dlf, df, duf, du2, ipiv, &fail);
  if (fail.code != NE_NOERROR)
    {
      printf("Error from nag_zgttrf (f07crc).\n%s\n", fail.message);
      exit_status = 1;
      goto END;
    }
  /* Solve the equations AX = B using nag_zgttrs (f07csc). */
  nag_zgttrs(order, Nag_NoTrans, n, nrhs, dlf, df, duf, du2, ipiv, x,
             pdx, &fail);
  if (fail.code != NE_NOERROR)
    {
      printf("Error from nag_zgttrs (f07csc).\n%s\n", fail.message);
      exit_status = 1;
      goto END;
    }

  /* Improve the solution and compute error estimates using
   * nag_zgtrfs (f07cvc).
   */
  nag_zgtrfs(order, Nag_NoTrans, n, nrhs, dl, d, du, dlf, df, duf, du2,
             ipiv, b, pdb, x, pdx, ferr, berr, &fail);
  if (fail.code != NE_NOERROR)
    {
      printf("Error from nag_zgtrfs (f07cvc).\n%s\n", fail.message);
      exit_status = 1;
      goto END;
    }

  /* Print the solution using nag_gen_complx_mat_print_comp (x04dbc). */
  fflush(stdout);
  nag_gen_complx_mat_print_comp(order, Nag_GeneralMatrix, Nag_NonUnitDiag, n,
                                nrhs, x, pdx, Nag_BracketForm, "%7.4f",
                                "Solution(s)", Nag_IntegerLabels, 0,
                                Nag_IntegerLabels, 0, 80, 0, 0, &fail);
  if (fail.code != NE_NOERROR)
    {
      printf("Error from nag_gen_complx_mat_print_comp (x04dbc).\n%s\n",
             fail.message);
      exit_status = 1;
      goto END;
    }

  /* Print the forward and backward error estimates */
  printf("\nBackward errors (machine-dependent)\n");
  for (j = 0; j < nrhs; ++j) printf("%11.1e%s", berr[j], j%7 == 6?"\n":" ");
```

```
  printf("\n\nEstimated forward error bounds (machine-dependent)\n");
  for (j = 0; j < nrhs; ++j) printf("%11.1e%s", ferr[j], j%7 == 6?"\n":" ");
  printf("\n");
 END:
  NAG_FREE(b);
  NAG_FREE(berr);
  NAG_FREE(d);
  NAG_FREE(df);
  NAG_FREE(dl);
  NAG_FREE(dlf);
  NAG_FREE(du);
  NAG_FREE(du2);
  NAG_FREE(duf);
  NAG_FREE(ferr);
  NAG_FREE(x);
  NAG_FREE(ipiv);

  return exit_status;
}

#undef B
```

## 10.2 Program Data

```
nag_zgtrfs (f07cvc) Example Program Data
    5            2                                                      : n, nrhs
               (  2.0, -1.0) (  2.0,  1.0) ( -1.0,  1.0) (  1.0, -1.0) : du
 ( -1.3,  1.3) ( -1.3,  1.3) ( -1.3,  3.3) ( -0.3,  4.3) ( -3.3,  1.3) : d
 (  1.0, -2.0) (  1.0 , 1.0) (  2.0, -3.0) (  1.0,  1.0)               : dl
 (  2.4, -5.0) (  2.7,  6.9)
 (  3.4, 18.2) ( -6.9, -5.3)
 (-14.7,  9.7) ( -6.0, -0.6)
 ( 31.9, -7.7) ( -3.9,  9.3)
 ( -1.0,  1.6) ( -3.0, 12.2)                                           : B
```

## 10.3 Program Results

```
nag_zgtrfs (f07cvc) Example Program Results

 Solution(s)
                     1                   2
 1  ( 1.0000, 1.0000)  ( 2.0000,-1.0000)
 2  ( 3.0000,-1.0000)  ( 1.0000, 2.0000)
 3  ( 4.0000, 5.0000)  (-1.0000, 1.0000)
 4  (-1.0000,-2.0000)  ( 2.0000, 1.0000)
 5  ( 1.0000,-1.0000)  ( 2.0000,-2.0000)

Backward errors (machine-dependent)
    3.7e-17     6.7e-17

Estimated forward error bounds (machine-dependent)
    5.4e-14     7.3e-14
```