NAG Library Function Document

nag_matop_real_gen_matrix_frcht_exp (f01jhc)

1 Purpose

nag_matop_real_gen_matrix_frcht_exp (f01jhc) computes the Fréchet derivative $L(A,E)$ of the matrix exponential of a real n by n matrix A applied to the real n by n matrix E. The matrix exponential e^A is also returned.

2 Specification

```
#include <nag.h>
#include <nagf01.h>
```

```
void nag_matop_real_gen_matrix_frcht_exp (Integer n, double a[],
     Integer pda, double e[], Integer pde, NagError *fail)
```

3 Description

The Fréchet derivative of the matrix exponential of A is the unique linear mapping $E \mapsto L(A,E)$ such that for any matrix E

$$e^{A+E} - e^A - L(A,E) = o(\|E\|).$$

The derivative describes the first-order effect of perturbations in A on the exponential e^A.

nag_matop_real_gen_matrix_frcht_exp (f01jhc) uses the algorithms of Al–Mohy and Higham (2009a) and Al–Mohy and Higham (2009b) to compute e^A and $L(A,E)$. The matrix exponential e^A is computed using a Padé approximant and the scaling and squaring method. The Padé approximant is then differentiated in order to obtain the Fréchet derivative $L(A,E)$.

4 References

Al–Mohy A H and Higham N J (2009a) A new scaling and squaring algorithm for the matrix exponential *SIAM J. Matrix Anal.* **31(3)** 970–989

Al–Mohy A H and Higham N J (2009b) Computing the Fréchet derivative of the matrix exponential, with an application to condition number estimation *SIAM J. Matrix Anal. Appl.* **30(4)** 1639–1657

Higham N J (2008) *Functions of Matrices: Theory and Computation* SIAM, Philadelphia, PA, USA

Moler C B and Van Loan C F (2003) Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later *SIAM Rev.* **45** 3–49

5 Arguments

1: **n** – Integer *Input*

On entry: n, the order of the matrix A.

Constraint: **n** ≥ 0.

2: **a**[*dim*] – double *Input/Output*

Note: the dimension, *dim*, of the array **a** must be at least **pda** × **n**.

The (i,j)th element of the matrix A is stored in **a**$[(j-1) \times$ **pda** $+ i - 1]$.

On entry: the n by n matrix A.

On exit: the n by n matrix exponential e^A.

3: **pda** – Integer *Input*

On entry: the stride separating matrix row elements in the array **a**.

Constraint: **pda** $\geq$ **n**.

4: **e**$[dim]$ – double *Input/Output*

Note: the dimension, *dim*, of the array **e** must be at least **pde** × **n**.

The (i, j)th element of the matrix E is stored in **e**$[(j - 1) \times$ **pde** $+ i - 1]$.

On entry: the n by n matrix E

On exit: the Fréchet derivative $L(A, E)$

5: **pde** – Integer *Input*

On entry: the stride separating matrix row elements in the array **e**.

Constraint: **pde** $\geq$ **n**.

6: **fail** – NagError * *Input/Output*

The NAG error argument (see Section 3.6 in the Essential Introduction).

6 Error Indicators and Warnings

NE_ALLOC_FAIL

Dynamic memory allocation failed.

NE_BAD_PARAM

On entry, argument ⟨*value*⟩ had an illegal value.

NE_INT

On entry, **n** = ⟨*value*⟩.
Constraint: **n** ≥ 0.

NE_INT_2

On entry, **pda** = ⟨*value*⟩ and **n** = ⟨*value*⟩.
Constraint: **pda** $\geq$ **n**.

On entry, **pde** = ⟨*value*⟩ and **n** = ⟨*value*⟩.
Constraint: **pde** $\geq$ **n**.

NE_INTERNAL_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

NE_SINGULAR

The linear equations to be solved for the Padé approximant are singular; it is likely that this function has been called incorrectly.

NW_SOME_PRECISION_LOSS

e^A has been computed using an IEEE double precision Padé approximant, although the arithmetic precision is higher than IEEE double precision.

7 Accuracy

For a normal matrix A (for which $A^{\mathrm{T}}A = AA^{\mathrm{T}}$) the computed matrix, e^A, is guaranteed to be close to the exact matrix, that is, the method is forward stable. No such guarantee can be given for non-normal matrices. See Section 10.3 of Higham (2008), Al–Mohy and Higham (2009a) and Al–Mohy and Higham (2009b) for details and further discussion.

8 Parallelism and Performance

nag_matop_real_gen_matrix_frcht_exp (f01jhc) is threaded by NAG for parallel execution in multithreaded implementations of the NAG Library.

nag_matop_real_gen_matrix_frcht_exp (f01jhc) makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the Users' Note for your implementation for any additional implementation-specific information.

9 Further Comments

The cost of the algorithm is $O(n^3)$ and the real allocatable memory required is approximately $9n^2$; see Al–Mohy and Higham (2009a) and Al–Mohy and Higham (2009b).

If the matrix exponential alone is required, without the Fréchet derivative, then nag_real_gen_matrix_exp (f01ecc) should be used.

If the condition number of the matrix exponential is required then nag_matop_real_gen_matrix_cond_exp (f01jgc) should be used.

As well as the excellent book Higham (2008), the classic reference for the computation of the matrix exponential is Moler and Van Loan (2003).

10 Example

This example finds the matrix exponential e^A and the Fréchet derivative $L(A, E)$, where

$$A = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 3 & 1 & 1 & 2 \\ 3 & 2 & 1 & 2 \\ 3 & 3 & 3 & 1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 4 & 2 & 1 & 2 \\ 0 & 3 & 2 & 1 \end{pmatrix}.$$

10.1 Program Text

```
/* nag_matop_real_gen_matrix_frcht_exp (f01jhc) Example Program.
 *
 * Copyright 2013 Numerical Algorithms Group.
 *
 * Mark 24, 2013.
 */
#include <nag.h>
#include <nag_stdlib.h>
#include <nagf01.h>
#include <nagx04.h>

#define A(I,J) a[J*pda + I]
#define E(I,J) e[J*pde + I]

int main(void)
{
  /* Scalars */
  Integer       exit_status = 0;
  Integer       i, j, n;
  Integer       pda, pde;
  /* Arrays */
  double        *a = 0;
  double        *e = 0;
  /* Nag Types */
  Nag_OrderType order = Nag_ColMajor;
  NagError      fail;

  INIT_FAIL(fail);

  /* Output preamble */
  printf("nag_matop_real_gen_matrix_frcht_exp (f01jhc) ");
  printf("Example Program Results\n\n");

  /* Skip heading in data file */
  scanf("%*[^\n] ");

  /* Read in the problem size */
  scanf("%ld%*[^\n] ", &n);

  pda = n;
  if (!(a = NAG_ALLOC(pda*n, double))) {
    printf("Allocation failure\n");
    exit_status = -1;
    goto END;
  }
  pde = n;
  if (!(e = NAG_ALLOC(pde*n, double))) {
    printf("Allocation failure\n");
    exit_status = -2;
    goto END;
  }

  /* Read in the matrix A from data file */
  for (i = 0; i < n; i++)
    for (j = 0; j < n; j++)
      scanf("%lf", &A(i, j));
  scanf("%*[^\n] ");

  /* Read in the matrix E from data file */
  for (i = 0; i < n; i++)
    for (j = 0; j < n; j++)
      scanf("%lf", &E(i, j));
  scanf("%*[^\n] ");

  /* Find exp(A) and L(A,E) using
   * nag_matop_real_gen_matrix_frcht_exp (f01jhc)
   * Frechet derivative of real matrix exponential
   */
  nag_matop_real_gen_matrix_frcht_exp(n, a, pda, e, pde, &fail);
  if (fail.code != NE_NOERROR) {
    printf("Error from nag_matop_real_gen_matrix_frcht_exp (f01jhc)\n%s\n",
           fail.message);
    exit_status = 1;
    goto END;
  }

  /* Print matrix exp(A) using nag_gen_real_mat_print (x04cac)
   * Print real general matrix (easy-to-use)
   */
  nag_gen_real_mat_print(order, Nag_GeneralMatrix, Nag_NonUnitDiag, n, n,
                         a, pda, "exp(A)", 0, &fail);
  if (fail.code != NE_NOERROR) {
    printf("Error from nag_gen_real_mat_print (x04cac)\n%s\n", fail.message);
    exit_status = 2;
  }

  /* Print matrix L(A,E) using nag_gen_real_mat_print (x04cac)
   * Print real general matrix (easy-to-use)
   */
  nag_gen_real_mat_print(order, Nag_GeneralMatrix, Nag_NonUnitDiag, n, n,
                         e, pde, "L(A,E)", 0, &fail);
  if (fail.code != NE_NOERROR) {
    printf("Error from nag_gen_real_mat_print (x04cac)\n%s\n", fail.message);
    exit_status = 3;
  }

 END:
  NAG_FREE(a);
  NAG_FREE(e);

  return exit_status;
}
```

10.2 Program Data

```
nag_matop_real_gen_matrix_frcht_exp (f01jhc) Example Program Data

  4                       :Value of n

  1.0   2.0   2.0   2.0
  3.0   1.0   1.0   2.0
  3.0   2.0   1.0   2.0
  3.0   3.0   3.0   1.0  :End of matrix a

  1.0   0.0   1.0   2.0
  0.0   0.0   0.0   1.0
  4.0   2.0   1.0   2.0
  0.0   3.0   2.0   1.0  :End of matrix e
```

10.3 Program Results

```
nag_matop_real_gen_matrix_frcht_exp (f01jhc) Example Program Results

 exp(A)
             1           2           3           4
 1    740.7038    610.8500    542.2743    549.1753
 2    731.2510    603.5524    535.0884    542.2743
 3    823.7630    679.4257    603.5524    610.8500
 4    998.4355    823.7630    731.2510    740.7038
 L(A,E)
             1           2           3           4
 1   3571.5724   2989.2581   2652.3449   2818.7416
 2   3202.0590   2684.2631   2381.4500   2542.7976
 3   4341.3950   3628.9329   3219.3516   3408.1831
 4   4821.2945   4035.9700   3580.0124   3804.4690
```