# NAG Toolbox

# nag_correg_mixeff_hier_init (g02jc)

## 1 Purpose

nag_correg_mixeff_hier_init (g02jc) preprocesses a dataset prior to fitting a linear mixed effects regression model of the following form via either nag_correg_mixeff_hier_reml (g02jd) or nag_correg_mixeff_hier_ml (g02je).

## 2 Syntax

```
[nff, nlsv, nrf, rcomm, icomm, ifail] = nag_correg_mixeff_hier_init(dat, levels,
y, fixed, rndm, lrcomm, licomm, 'n', n, 'ncol', ncol, 'wt', wt, 'lfixed',
lfixed, 'nrndm', nrndm)
```

```
[nff, nlsv, nrf, rcomm, icomm, ifail] = g02jc(dat, levels, y, fixed, rndm,
lrcomm, licomm, 'n', n, 'ncol', ncol, 'wt', wt, 'lfixed', lfixed, 'nrndm',
nrndm)
```

**Note**: the interface to this routine has changed since earlier releases of the toolbox:

At Mark 23: *weight* was removed from the interface.

## 3 Description

nag_correg_mixeff_hier_init (g02jc) must be called prior to fitting a linear mixed effects regression model with either nag_correg_mixeff_hier_reml (g02jd) or nag_correg_mixeff_hier_ml (g02je).

The model fitting functions nag_correg_mixeff_hier_reml (g02jd) and nag_correg_mixeff_hier_ml (g02je) fit a model of the following form:

$$y = X\beta + Z\nu + \epsilon$$

where $y$ is a vector of $n$ observations on the dependent variable,

$X$ is an $n$ by $p$ design matrix of *fixed* independent variables,

$\beta$ is a vector of $p$ unknown *fixed effects*,

$Z$ is an $n$ by $q$ design matrix of *random* independent variables,

$\nu$ is a vector of length $q$ of unknown *random effects*,

$\epsilon$ is a vector of length $n$ of unknown random errors,

and $\nu$ and $\epsilon$ are Normally distributed with expectation zero and variance/covariance matrix defined by

$$\mathrm{Var}\begin{bmatrix} \nu \\ \epsilon \end{bmatrix} = \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix}$$

where $R = \sigma_R^2 I$, $I$ is the $n \times n$ identity matrix and $G$ is a diagonal matrix.

Case weights can be incorporated into the model by replacing $X$ and $Z$ with $W_c^{1/2}X$ and $W_c^{1/2}Z$ respectively where $W_c$ is a diagonal weight matrix.

## 4 References

None.

# 5 Parameters

## 5.1 Compulsory Input Parameters

1: **dat**($lddat$, **ncol**) – REAL (KIND=nag_wp) array

$lddat$, the first dimension of the array, must satisfy the constraint $lddat \geq$ **n**.

A matrix of data, with **dat**$(i, j)$ holding the $i$th observation on the $j$th variable. The two design matrices $X$ and $Z$ are constructed from **dat** and the information given in **fixed** (for $X$) and **rndm** (for $Z$).

*Constraint*: if **levels**$(j) \neq 1, 1 \leq$ **dat**$(i, j) \leq$ **levels**$(j)$.

2: **levels**(**ncol**) – INTEGER array

**levels**$(i)$ contains the number of levels associated with the $i$th variable held in **dat**.

If the $i$th variable is continuous or binary (i.e., only takes the values zero or one) then **levels**$(i)$ must be set to 1. Otherwise the $i$th variable is assumed to take an integer value between 1 and **levels**$(i)$, (i.e., the $i$th variable is discrete with **levels**$(i)$ levels).

*Constraint*: **levels**$(i) \geq 1$, for $i = 1, 2, \ldots,$ **ncol**.

3: **y**(**n**) – REAL (KIND=nag_wp) array

$y$, the vector of observations on the dependent variable.

4: **fixed**(**lfixed**) – INTEGER array

Defines the structure of the fixed effects design matrix, $X$.

**fixed**$(1)$
: The number of variables, $N_F$, to include as fixed effects (not including the intercept if present).

**fixed**$(2)$
: The fixed intercept flag which must contain 1 if a fixed intercept is to be included and 0 otherwise.

**fixed**$(2 + i)$
: The column of **dat** holding the $i$th fixed variable, for $i = 1, 2, \ldots,$ **fixed**$(1)$.

See Section 9.1 for more details on the construction of $X$.

*Constraints*:

**fixed**$(1) \geq 0$;
**fixed**$(2) = 0$ or 1;
$1 \leq$ **fixed**$(2 + i) \leq$ **ncol**, for $i = 1, 2, \ldots,$ **fixed**$(1)$.

5: **rndm**($ldrndm$, **nrndm**) – INTEGER array

$ldrndm$, the first dimension of the array, must satisfy the constraint $ldrndm \geq \max_b(3 + N_{R_b} + N_{S_b})$.

**rndm**$(i, j)$ defines the structure of the *random effects* design matrix, $Z$. The $b$th column of **rndm** defines a block of columns in the design matrix $Z$.

**rndm**$(1, b)$
: The number of variables, $N_{R_b}$, to include as random effects in the $b$th block (not including the random intercept if present).

**rndm**$(2, b)$
: The random intercept flag which must contain 1 if block $b$ includes a random intercept and 0 otherwise.

**rndm**$(2 + i, b)$
The column of **dat** holding the $i$th random variable in the $b$th block, for $i = 1, 2, \ldots,$ **rndm**$(1, b)$.

**rndm**$(3 + N_{R_b}, b)$
The number of subject variables, $N_{S_b}$, for the $b$th block. The subject variables define the nesting structure for this block.

**rndm**$(3 + N_{R_b} + i, b)$
The column of **dat** holding the $i$th subject variable in the $b$th block, for $i = 1, 2, \ldots,$ **rndm**$(3 + N_{R_b}, b)$.

See Section 9.2 for more details on the construction of $Z$.

*Constraints*:

**rndm**$(1, b) \geq 0$;
**rndm**$(2, b) = 0$ or 1;
at least one random variable or random intercept must be specified in each block, i.e., **rndm**$(1, b) +$ **rndm**$(2, b) > 0$;
the column identifiers associated with the random variables must be in the range 1 to **ncol**, i.e., $1 \leq$ **rndm**$(2 + i, b) \leq$ **ncol**, for $i = 1, 2, \ldots,$ **rndm**$(1, b)$;
**rndm**$(3 + N_{R_b}, b) \geq 0$;
the column identifiers associated with the subject variables must be in the range 1 to **ncol**, i.e., $1 \leq$ **rndm**$(3 + N_{R_b} + i, b) \leq$ **ncol**, for $i = 1, 2, \ldots,$ **rndm**$(3 + N_{R_b}, b)$.

6: **lrcomm** – INTEGER

The dimension of the array **rcomm**.

*Constraint*: **lrcomm** $\geq$ **nrf** $\times$ **nlsv** $+$ **nff** $+$ **nff** $\times$ **nlsv** $+$ **nrf** $\times$ **nlsv** $+$ **nff** $+ 2$.

7: **licomm** – INTEGER

The dimension of the array **icomm**.

*Constraint*: **licomm** $= 2$ or
**licomm** $\geq 34 + N_F \times (\mathrm{MFL} + 1) +$ **nrndm** $\times \mathrm{MNR} \times \mathrm{MRL} + (\mathrm{LRNDM} + 2) \times$ **nrndm** $+$ **ncol** $+ \mathrm{LDID} \times \mathrm{LB}$,

where

$\mathrm{MNR} = \max_b(N_{R_b})$,

$\mathrm{MFL} = \max_i($**levels**$($**fixed**$(2 + i)))$,

$\mathrm{MRL} = \max_{b,i}($**levels**$($**rndm**$(2 + i, b)))$,

$\mathrm{LDID} = \max_b N_{S_b}$,

$\mathrm{LB} =$ **nff** $+$ **nrf** $\times$ **nlsv**, and

$\mathrm{LRNDM} = \underset{b}{max}(3 + N_{R_b} + N_{S_b})$

## 5.2 Optional Input Parameters

1: **n** – INTEGER

*Default*: the dimension of the array **y** and the first dimension of the array **dat**. (An error is raised if these dimensions are not equal.)

$n$, the number of observations.

The effective number of observations, that is the number of observations with nonzero weight (see **wt** for more detail), must be greater than the number of fixed effects in the model (as returned in **nff**).

*Constraint*: $\mathbf{n} \geq 1$.

2: **ncol** – INTEGER

*Default*: the dimension of the arrays **dat**, **levels** and the second dimension of the array **dat**. (An error is raised if these dimensions are not equal.)

The number of columns in the data matrix, **dat**.

*Constraint*: $\mathbf{ncol} \geq 0$.

3: **wt**(:) – REAL (KIND=nag_wp) array

The dimension of the array **wt** must be at least **n** if $weight =$ 'W'

If $weight =$ 'W', **wt** must contain the diagonal elements of the weight matrix $W_c$.

If $\mathbf{wt}(i) = 0.0$, the $i$th observation is not included in the model and the effective number of observations is the number of observations with nonzero weights.

If $weight =$ 'U', **wt** is not referenced and the effective number of observations is $n$.

*Constraint*: if $weight =$ 'W', $\mathbf{wt}(i) \geq 0.0$, for $i = 1, 2, \ldots, n$.

4: **lfixed** – INTEGER

*Default*: the dimension of the array **fixed**.

Length of the vector **fixed**.

*Constraint*: $\mathbf{lfixed} \geq 2 + \mathbf{fixed}(1)$.

5: **nrndm** – INTEGER

*Default*: the second dimension of the array **rndm**.

The second dimension of the array **rndm**.

*Constraint*: $\mathbf{nrndm} > 0$.

## 5.3 Output Parameters

1: **nff** – INTEGER

$p$, the number of fixed effects estimated, i.e., the number of columns in the design matrix $X$.

2: **nlsv** – INTEGER

The number of levels for the overall subject variable (see Section 9.2 for a description of what this means). If there is no overall subject variable, $\mathbf{nlsv} = 1$.

3: **nrf** – INTEGER

The number of random effects estimated in each of the overall subject blocks. The number of columns in the design matrix $Z$ is given by $q = \mathbf{nrf} \times \mathbf{nlsv}$.

4: **rcomm**(**lrcomm**) – REAL (KIND=nag_wp) array

Communication array as required by the analysis functions nag_correg_mixeff_hier_reml (g02jd) and nag_correg_mixeff_hier_ml (g02je).

5: **icomm**(**licomm**) – INTEGER array

If **licomm** $= 2$, **icomm**$(1)$ holds the minimum required value for **licomm** and **icomm**$(2)$ holds the minimum required value for **lrcomm**, otherwise **icomm** is a communication array as required by the analysis functions nag_correg_mixeff_hier_reml (g02jd) and nag_correg_mixeff_hier_ml (g02je).

6: **ifail** – INTEGER

**ifail** $= 0$ unless the function detects an error (see Section 5).

## 6 Error Indicators and Warnings

Errors or warnings detected by the function:

**ifail** $= 1$

On entry, $weight$ had an illegal value.

**ifail** $= 2$

Constraint: **n** $\geq 1$.

**ifail** $= 3$

Constraint: **ncol** $\geq 0$.

**ifail** $= 4$

On entry, variable $j$ of observation $i$ is less than 1 or greater than **levels**$(j)$.

**ifail** $= 5$

Constraint: $lddat \geq$ **n**.

**ifail** $= 6$

Constraint: **levels**$(i) \geq 1$.

**ifail** $= 8$

Constraint: **wt**$(i) \geq 0.0$.

**ifail** $= 9$

On entry, number of fixed parameters, $\langle$*value*$\rangle$ is less than zero.

**ifail** $= 10$

**lfixed** is too small.

**ifail** $= 11$

Constraint: **nrndm** $> 0$.

**ifail** $= 12$

On entry, number of random parameters for random statement $i$ is less than 0.

**ifail** $= 13$

$ldrndm$ is too small.

**ifail** $= 18$

**lrcomm** is too small.

**ifail** $= 20$

**licomm** is too small.

**ifail** $= 102$

**n** is too small.

**ifail** $= 108$

On entry, no observations due to zero weights.

**ifail** $= 109$

On entry, invalid value for fixed intercept flag.

**ifail** $= 112$

On entry, invalid value for random intercept flag for random statement $i$.

**ifail** $= 209$

On entry, index of fixed variable $j$ is less than 1 or greater than **ncol**.

**ifail** $= 212$

On entry, must be at least one parameter, or an intercept in each random statement $i$:

**ifail** $= 312$

On entry, index of random variable $j$ in random statement $i$ is less than 1 or greater than **ncol**.

**ifail** $= 412$

On entry, number of subject parameters for random statement $i$ is less than 0.

**ifail** $= 512$

On entry, nesting variable $j$ in random statement $i$ has one level.

**ifail** $= -99$

An unexpected error has been triggered by this routine. Please contact NAG.

**ifail** $= -399$

Your licence key may have expired or may not have been installed correctly.

**ifail** $= -999$

Dynamic memory allocation failed.

# 7 Accuracy

Not applicable.

# 8 Further Comments

## 8.1 Construction of the *fixed effects* design matrix, *X*

Let

$N_F$ denote the number of fixed variables, that is **fixed**$(1) = N_F$;

$F_j$ denote the $j$th fixed variable, that is the vector of values held in the $k$th column of **dat** when **fixed**$(2 + j) = k$;

$F_{ij}$ denote the $i$th element of $F_j$;

$L(F_j)$ denote the number of levels for $F_j$, that is $L(F_j) = \textbf{levels}(\textbf{fixed}(2+j))$;

$D_v(F_j)$ denoted an indicator function that returns a vector of values whose $i$th element is 1 if $F_{ij} = v$ and 0 otherwise.

The design matrix for the *fixed effects*, $X$, is constructed as follows:

set $k$ to one and the flag done_first to false;

if a fixed intercept is included, that is $\textbf{fixed}(2) = 1$,

set the first column of $X$ to a vector of 1s;

set $k = k + 1$;

set done_first to true;

loop over each fixed variable, so for each $j = 1, 2, \dots, N_F$,

if $L(F_j) = 1$,

set the $k$th column of $X$ to be $F_j$;

set $k = k + 1$;

else

if done_first is false then

set the $L(F_j)$ columns, $k$ to $k + L(F_j) - 1$, of $X$ to $D_v(F_j)$, for $v = 1, 2, \dots, L(F_j)$;

set $k = k + L(F_j)$;

set done_first to true;

else

set the $L(F_j) - 1$ columns, $k$ to $k + L(F_j) - 2$, of $X$ to $D_v(F_j)$, for $v = 2, 3, \dots, L(F_j)$;

set $k = k + L(F_j) - 1$.

The number of columns in the design matrix, $X$, is therefore given by

$$p = 1 + \sum_{j=1}^{N_F} (\textbf{levels}(\textbf{fixed}(2+j)) - 1).$$

This quantity is returned in **nff**.

In summary, nag_correg_mixeff_hier_init (g02jc) converts all non-binary categorical variables (i.e., where $L(F_j) > 1$) to dummy variables. If a fixed intercept is included in the model then the first level of all such variables is dropped. If a fixed intercept is not included in the model then the first level of all such variables, other than the first, is dropped. The variables are added into the model in the order they are specified in **fixed**.

## 8.2 Construction of *random effects* design matrix, *Z*

Let

$N_{R_b}$ denote the number of random variables in the $b$th random statement, that is $N_{R_b} = \textbf{rndm}(1, b)$;

$R_{jb}$ denote the $j$th random variable from the $b$th random statement, that is the vector of values held in the $k$th column of **dat** when $\textbf{rndm}(2+j, b) = k$;

$R_{ijb}$ denote the $i$th element of $R_{jb}$;

$L(R_{jb})$ denote the number of levels for $R_{jb}$, that is $L(R_{jb}) = \textbf{levels}(\textbf{rndm}(2+j, b))$;

$D_v(R_{jb})$ denoted an indicator function that returns a vector of values whose $i$th element is 1 if $R_{ijb} = v$ and 0 otherwise;

$N_{S_b}$ denote the number of subject variables in the $b$th random statement, that is $N_{S_b} = \mathbf{rndm}(3 + N_{R_b}, b)$;

$S_{jb}$ denote the $j$th subject variable from the $b$th random statement, that is the vector of values held in the $k$th column of **dat** when $\mathbf{rndm}(3 + N_{R_b} + j, b) = k$;

$S_{ijb}$ denote the $i$th element of $S_{jb}$;

$L(S_{jb})$ denote the number of levels for $S_{jb}$, that is $L(S_{jb}) = \mathbf{levels}(\mathbf{rndm}(3 + N_{R_b} + j, b))$;

$I_b\left(s_1, s_2, \ldots, s_{N_{S_b}}\right)$ denoted an indicator function that returns a vector of values whose $i$th element is 1 if $S_{ijb} = s_j$ for all $j = 1, 2, \ldots, N_{S_b}$ and 0 otherwise.

The design matrix for the *random effects*, $Z$, is constructed as follows:

set $k$ to one;

loop over each random statement, so for each $b = 1, 2, \ldots, \mathbf{nrndm}$,

loop over each level of the last subject variable, so for each $s_{N_{S_b}} = 1, 2, \ldots, L\left(R_{N_{S_b} b}\right)$,

$\vdots$

loop over each level of the second subject variable, so for each $s_2 = 1, 2, \ldots, L(R_{2b})$,

loop over each level of the first subject variable, so for each $s_1 = 1, 2, \ldots, L(R_{1b})$,

if a random intercept is included, that is $\mathbf{rndm}(2, b) = 1$,

set the $k$th column of $Z$ to $I_b\left(s_1, s_2, \ldots, s_{N_{S_b}}\right)$;

set $k = k + 1$;

loop over each random variable in the $b$th random statement, so for each $j = 1, 2, \ldots, N_{R_b}$,

if $L(R_{jb}) = 1$,

set the $k$th column of $Z$ to $R_{jb} \times I_b\left(s_1, s_2, \ldots, s_{N_{S_b}}\right)$ where $\times$ indicates an element-wise multiplication between the two vectors, $R_{jb}$ and $I_b(\ldots)$;

set $k = k + 1$;

else

set the $L(R_{bj})$ columns, $k$ to $k + L(R_{bj})$, of $Z$ to $D_v(R_{jb}) \times I_b\left(s_1, s_2, \ldots, s_{N_{S_b}}\right)$, for $v = 1, 2, \ldots, L(R_{jb})$. As before, $\times$ indicates an element-wise multiplication between the two vectors, $D_v(\ldots)$ and $I_b(\ldots)$;

set $k = k + L(R_{jb})$.

In summary, each column of **rndm** defines a block of consecutive columns in $Z$. nag_correg_mixeff_hier_init (g02jc) converts all non-binary categorical variables (i.e., where $L(R_{jb})$ or $L(S_{jb}) > 1$) to dummy variables. All random variables defined within a column of **rndm** are nested within all subject variables defined in the same column of **rndm**. In addition each of the subject variables are nested within each other, starting with the first (i.e., each of the $R_{jb}, j = 1, 2, \ldots, N_{R_b}$ are nested within $S_{1b}$ which in turn is nested within $S_{2b}$, which in turn is nested within $S_{3b}$, etc.).

If the last subject variable in each column of **rndm** are the same (i.e., $S_{N_{S_1} 1} = S_{N_{S_2} 2} = \ldots = S_{N_{S_b} b}$) then all random effects in the model are nested within this variable. In such instances the last subject variable ($S_{N_{S_1} 1}$) is called the overall subject variable. The fact that all of the random effects in the model are nested within the overall subject variable means that $Z^{\mathrm{T}} Z$ is block diagonal in structure. This fact can be utilised to improve the efficiency of the underlying computation and reduce the amount of internal storage required. The number of levels in the overall subject variable is returned in $\mathbf{nlsv} = L\left(S_{N_{S_1}1}\right)$.

If the last $k$ subject variables in each column of **rndm** are the same, for $k > 1$ then the overall subject variable is defined as the interaction of these $k$ variables and

$$\mathbf{nlsv} = \prod_{j=N_{S_1}-k+1}^{N_{S_1}} L(S_{j1}).$$

If there is no overall subject variable then $\mathbf{nlsv} = 1$.

The number of columns in the design matrix $Z$ is given by $q = \mathbf{nrf} \times \mathbf{nlsv}$.

## 8.3 The rndm argument

To illustrate some additional points about the **rndm** argument, we assume that we have a dataset with three discrete variables, $V_1$, $V_2$ and $V_3$, with $2, 4$ and $3$ levels respectively, and that $V_1$ is in the first column of **dat**, $V_2$ in the second and $V_3$ the third. Also assume that we wish to fit a model containing $V_1$ along with $V_2$ nested within $V_3$, as random effects. In order to do this the **rndm** matrix requires two columns:

$$\mathbf{rndm} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 2 \\ 0 & 1 \\ 0 & 3 \end{pmatrix}$$

The first column, $(1, 0, 1, 0, 0)$, indicates one random variable ($\mathbf{rndm}(1,1) = 1$), no intercept ($\mathbf{rndm}(2,1) = 0$), the random variable is in the first column of **dat** ($\mathbf{rndm}(3,1) = 1$), there are no subject variables; as no nesting is required for $V_1$ ($\mathbf{rndm}(4,1) = 0$). The last element in this column is ignored.

The second column, $(1, 0, 2, 1, 3)$, indicates one random variable ($\mathbf{rndm}(1,2) = 1$), no intercept ($\mathbf{rndm}(2,2) = 0$), the random variable is in the second column of **dat** ($\mathbf{rndm}(3,2) = 2$), there is one subject variable ($\mathbf{rndm}(4,2) = 1$), and the subject variable is in the third column of **dat** ($\mathbf{rndm}(5,2) = 3$).

The corresponding $Z$ matrix would have 14 columns, with 2 coming from $V_1$ and 12 ($4 \times 3$) from $V_2$ nested within $V_3$. The, symmetric, $Z^{\mathrm{T}}Z$ matrix has the form

$$\begin{pmatrix}
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & 0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 \\
- & - & 0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 \\
- & - & 0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 \\
- & - & 0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 \\
- & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\
- & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\
- & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\
- & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & -
\end{pmatrix}$$

where 0 indicates a structural zero, i.e., it always takes the value 0, irrespective of the data, and $-$ a value that is not a structural zero. The first two rows and columns of $Z^{\mathrm{T}}Z$ correspond to $V_1$. The block diagonal matrix in the 12 rows and columns in the bottom right correspond to $V_2$ nested within $V_3$. With the $4 \times 4$ blocks corresponding to the levels of $V_2$. There are three blocks as the subject variable ($V_3$) has three levels.

The model fitting functions, nag_correg_mixeff_hier_reml (g02jd) and nag_correg_mixeff_hier_ml (g02je), use the sweep algorithm to calculate the log-likelihood function for a given set of variance components. This algorithm consists of moving down the diagonal elements (called pivots) of a matrix which is similar in structure to $Z^{\mathrm{T}}Z$, and updating each element in that matrix. When using the $k$ diagonal element of a matrix $A$, an element $a_{ij}, i \neq k, j \neq k$, is adjusted by an amount equal to $a_{ik}a_{ij}/a_{kk}$. This process can be referred to as sweeping on the $k$th pivot. As there are no structural zeros in the first row or column of the above $Z^{\mathrm{T}}Z$, sweeping on the first pivot of $Z^{\mathrm{T}}Z$ would alter each element of the matrix and therefore destroy the structural zeros, i.e., we could no longer guarantee they would be zero.

Reordering the **rndm** matrix to

$$\mathbf{rndm} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 1 \\ 1 & 0 \\ 3 & 0 \end{pmatrix}$$

i.e., the swapping the two columns, results in a $Z^{\mathrm{T}}Z$ matrix of the form

$$\begin{pmatrix}
- & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - \\
- & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - \\
- & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - \\
- & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - \\
0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & - & - \\
0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & - & - \\
0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & - & - \\
0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & -
\end{pmatrix}$$

This matrix is identical to the previous one, except the first two rows and columns have become the last two rows and columns. Sweeping a matrix, $A = \{a_{ij}\}$, of this form on the first pivot will only affect those elements $a_{ij}$, where $a_{i1} \neq 0$ and $a_{1j} \neq 0$, which is only the 13th and 14th row and columns, and the top left hand block of 4 rows and columns. The block diagonal nature of the first 12 rows and columns therefore greatly reduces the amount of work the algorithm needs to perform.

nag_correg_mixeff_hier_init (g02jc) constructs the $Z^{\mathrm{T}}Z$ as specified by the **rndm** matrix, and does not attempt to reorder it to improve performance. Therefore for best performance some thought is required on what ordering to use. In general it is more efficient to structure **rndm** in such a way that the first row relates to the deepest level of nesting, the second to the next level, etc..

# 9 Example

See Section 10 in nag_correg_mixeff_hier_reml (g02jd) and nag_correg_mixeff_hier_ml (g02je).

## 9.1 Program Text

```
     function g02jc_example

fprintf('g02jc example results\n\n');

% Problem size
n = 90;
ncol = 12;
nrndm = 3;
nvpr = nag_int(7);

% The number of levels associated with each of the independent variables
levels = [nag_int(2); 3; 2; 3; 2; 3; 1; 4; 5; 2; 3; 3];

% The Fixed part of the model
```

```
fixed = [nag_int(2); 1; 1; 2];

% The Random part of the model
rndm = [nag_int(2), 2,  3;
                0,  0,  0;
                3,  5,  7;
                4,  6,  8;
                3,  2,  9;
               10, 11,  1;
               11, 12, 12;
               12,  0,  0];

% Dependant data
y = [ 3.1100;   2.8226;   7.4543;   4.4313;   6.1543;  -0.1783;
      4.6748;   7.0667;   1.4262;   7.7290;  -2.1806;   6.8419;
      1.2590;   8.8405;   6.1657;  -4.5605;  -1.2367; -12.2932;
     -2.3374;   0.0716;   0.1895;   1.5608;  -0.8529;  -4.1169;
      3.9977;  -8.1277;  -4.9656;  -0.6428;  -5.5152;  -5.5657;
     14.8177;  16.9783;  13.8966;  14.8166;  19.3640;   9.5299;
     12.0102;   6.1551;  -1.7048;   2.7640;   2.8065;   0.0974;
     -7.8080; -18.0450;  -2.8199;   8.9893;   3.7978;  -6.3493;
      8.1411;  -7.5483;  -0.4600;  -3.2135;  -6.6562;   5.1267;
      3.5592;  -4.4420;  -8.5965;  -6.3187;  -7.8953; -10.1383;
     -7.8850;  23.2001;   5.5829;  -4.3698;   2.1274;  -2.7184;
    -17.9128;  -1.2708; -24.2735; -14.7374;   0.1713;   8.0006;
      1.2100;   3.3307; -22.6713;   7.5562;  -7.0694;   3.7159;
     -4.3135; -14.5577; -12.5107;   4.7708;  13.2797;  -6.3243;
     -7.0549;  -9.2713; -18.7788;  -7.7230; -22.7230; -11.6609];

% Independent data
dat = [1, 3, 2, 1, 2, 2,-0.3160, 4, 2, 1, 1, 1;
       1, 1, 1, 3, 1, 2,-1.3377, 1, 4, 1, 1, 1;
       1, 3, 1, 3, 1, 3,-0.7610, 4, 2, 1, 1, 1;
       2, 3, 2, 1, 1, 3,-2.2976, 4, 2, 1, 1, 1;
       2, 2, 1, 3, 2, 3,-0.4263, 2, 1, 1, 1, 1;
       2, 1, 2, 3, 1, 3, 1.4067, 3, 3, 2, 1, 1;
       2, 3, 2, 1, 2, 1,-1.4669, 1, 2, 2, 1, 1;
       1, 1, 1, 3, 2, 3, 0.4717, 2, 4, 2, 1, 1;
       1, 3, 2, 3, 2, 1, 0.4436, 1, 3, 2, 1, 1;
       1, 1, 1, 2, 2, 3,-0.5950, 3, 4, 2, 1, 1;
       1, 3, 1, 3, 1, 1,-1.7981, 4, 2, 1, 2, 1;
       2, 3, 1, 2, 1, 1, 0.2397, 1, 4, 1, 2, 1;
       1, 2, 2, 1, 2, 3, 0.4742, 1, 1, 1, 2, 1;
       2, 2, 2, 2, 2, 3, 0.6888, 3, 1, 1, 2, 1;
       2, 1, 2, 3, 1, 3,-1.0616, 3, 5, 1, 2, 1;
       1, 2, 2, 2, 2, 1,-0.5356, 1, 3, 2, 2, 1;
       1, 3, 2, 2, 1, 1,-1.2963, 2, 5, 2, 2, 1;
       1, 2, 2, 1, 2, 2,-1.5389, 3, 2, 2, 2, 1;
       2, 3, 1, 1, 2, 2,-0.6408, 2, 1, 2, 2, 1;
       1, 2, 2, 2, 1, 1, 0.6574, 1, 1, 2, 2, 1;
       2, 1, 1, 1, 1, 3, 0.9259, 1, 2, 1, 3, 1;
       2, 2, 2, 1, 2, 2, 1.5080, 3, 1, 1, 3, 1;
       2, 3, 1, 1, 1, 3, 2.5821, 2, 3, 1, 3, 1;
       1, 2, 2, 1, 2, 3, 0.4102, 1, 4, 1, 3, 1;
       2, 1, 2, 3, 2, 2, 0.7839, 2, 5, 1, 3, 1;
       1, 2, 2, 3, 2, 1,-1.8812, 4, 2, 2, 3, 1;
       1, 2, 1, 3, 2, 3, 0.7770, 4, 1, 2, 3, 1;
       2, 2, 1, 2, 1, 3, 0.2590, 3, 1, 2, 3, 1;
       2, 3, 2, 2, 2, 3,-0.9250, 3, 3, 2, 3, 1;
       2, 2, 1, 3, 2, 3,-0.4831, 1, 5, 2, 3, 1;
       2, 2, 1, 3, 1, 3, 0.5046, 3, 3, 1, 1, 2;
       2, 1, 1, 2, 2, 1,-0.6903, 2, 1, 1, 1, 2;
       1, 3, 2, 2, 2, 1, 1.6166, 2, 5, 1, 1, 2;
       2, 2, 2, 2, 1, 3, 0.2778, 2, 3, 1, 1, 2;
       2, 3, 2, 2, 1, 2, 1.9586, 4, 2, 1, 1, 2;
       1, 3, 1, 1, 1, 3, 1.0506, 2, 5, 2, 1, 2;
       2, 1, 1, 3, 2, 3, 0.4871, 1, 1, 2, 1, 2;
       2, 1, 2, 3, 2, 1, 2.0891, 4, 4, 2, 1, 2;
       1, 2, 1, 1, 2, 2, 1.4338, 4, 3, 2, 1, 2;
       1, 1, 2, 3, 1, 2,-1.1196, 3, 4, 2, 1, 2;
       1, 3, 1, 1, 2, 1, 0.3367, 3, 2, 1, 2, 2;
```

```
       2, 2, 1, 3, 1, 1, 0.1092, 2, 2, 1, 2, 2;
       1, 1, 1, 2, 2, 2, 0.4007, 4, 1, 1, 2, 2;
       2, 3, 1, 1, 1, 2, 0.1460, 3, 5, 1, 2, 2;
       2, 1, 2, 3, 1, 3,-0.3877, 3, 4, 1, 2, 2;
       1, 1, 1, 2, 2, 1, 0.6957, 4, 3, 2, 2, 2;
       2, 1, 1, 1, 2, 1,-0.4664, 3, 3, 2, 2, 2;
       1, 1, 1, 1, 2, 3, 0.2067, 2, 4, 2, 2, 2;
       2, 1, 2, 1, 1, 2, 0.4112, 1, 4, 2, 2, 2;
       2, 2, 1, 1, 1, 2,-1.3734, 3, 3, 2, 2, 2;
       2, 1, 2, 3, 1, 3, 0.7065, 1, 3, 1, 3, 2;
       1, 2, 2, 2, 1, 2, 1.3628, 4, 2, 1, 3, 2;
       2, 1, 2, 2, 2, 3,-0.5052, 4, 5, 1, 3, 2;
       2, 1, 1, 1, 2, 1,-1.3457, 2, 5, 1, 3, 2;
       1, 1, 2, 1, 2, 3,-1.8022, 3, 4, 1, 3, 2;
       2, 3, 1, 2, 1, 1, 0.0116, 2, 4, 2, 3, 2;
       2, 2, 1, 3, 2, 3,-0.9075, 1, 3, 2, 3, 2;
       2, 2, 2, 2, 2, 3,-1.4707, 1, 1, 2, 3, 2;
       2, 2, 1, 1, 2, 1,-1.2938, 2, 3, 2, 3, 2;
       1, 3, 1, 3, 2, 2,-1.1660, 4, 4, 2, 3, 2;
       1, 2, 1, 1, 2, 3, 0.0397, 4, 4, 1, 1, 3;
       1, 3, 1, 2, 1, 3,-0.5987, 3, 2, 1, 1, 3;
       2, 3, 2, 2, 1, 1, 0.6683, 3, 3, 1, 1, 3;
       2, 2, 1, 1, 2, 2,-0.0106, 1, 3, 1, 1, 3;
       1, 2, 1, 3, 2, 2, 0.5885, 1, 3, 1, 1, 3;
       1, 1, 1, 1, 1, 2, 0.4555, 1, 5, 2, 1, 3;
       2, 2, 2, 1, 1, 2, 0.6502, 4, 3, 2, 1, 3;
       1, 1, 1, 3, 1, 1,-0.1601, 1, 3, 2, 1, 3;
       2, 2, 1, 3, 2, 3, 1.6910, 1, 1, 2, 1, 3;
       2, 2, 2, 3, 1, 2, 0.1053, 4, 4, 2, 1, 3;
       2, 1, 2, 3, 2, 2,-0.4037, 3, 4, 1, 2, 3;
       1, 3, 2, 3, 1, 3,-0.5853, 3, 2, 1, 2, 3;
       2, 3, 2, 1, 1, 1,-0.3037, 1, 3, 1, 2, 3;
       1, 3, 1, 1, 2, 2,-0.0774, 1, 4, 1, 2, 3;
       2, 3, 1, 2, 2, 1, 0.4733, 4, 5, 1, 2, 3;
       1, 3, 2, 2, 1, 2,-0.0354, 4, 2, 2, 2, 3;
       1, 3, 2, 2, 1, 1,-0.6640, 2, 1, 2, 2, 3;
       2, 3, 1, 3, 1, 1, 0.0335, 4, 4, 2, 2, 3;
       1, 2, 2, 2, 1, 3, 0.1351, 1, 1, 2, 2, 3;
       1, 1, 2, 1, 2, 3,-0.5951, 3, 4, 2, 2, 3;
       2, 2, 2, 3, 1, 3, 0.2735, 3, 2, 1, 3, 3;
       2, 2, 1, 1, 1, 3, 0.3157, 1, 2, 1, 3, 3;
       2, 2, 2, 1, 1, 1,-1.0843, 2, 3, 1, 3, 3;
       1, 2, 2, 1, 2, 2,-0.0836, 4, 2, 1, 3, 3;
       2, 1, 2, 1, 1, 2,-0.2884, 2, 1, 1, 3, 3;
       2, 3, 2, 3, 2, 3,-0.1006, 1, 2, 2, 3, 3;
       1, 3, 1, 2, 2, 3, 0.5710, 1, 3, 2, 3, 3;
       1, 1, 2, 1, 1, 2, 0.2776, 2, 3, 2, 3, 3;
       2, 3, 2, 2, 1, 3,-0.7561, 4, 4, 2, 3, 3;
       1, 2, 2, 2, 1, 2, 1.5549, 1, 4, 2, 3, 3];

vpr = [nag_int(1):7];
gamma = zeros(8, 1);
gamma(1) = -1; % Estimate initial values for the variance components

% Call the initialisation routine once to get lrcomm and licomm
lrcomm = nag_int(0);
licomm = nag_int(2);
[nff, nlsv, nrf, rcomm, icomm, ifail] = ...
  g02jc( ...
         dat, levels, y, fixed, rndm, lrcomm, licomm);
licomm = icomm(1);
lrcomm = icomm(2);

% Pre-process the data
[nff, nlsv, nrf, rcomm, icomm, ifail] = ...
  g02jc( ...
         dat, levels, y, fixed, rndm, lrcomm, licomm);

% Use default options
iopt = zeros(0, 0, nag_int_name);
ropt = zeros(0, 0);
```

```
lb = nff + nrf*nlsv;

% Perform the analysis
[gamma, effn, rnkx, ncov, lnlike, id, b, se, czz, cxx, cxz, ifail] = ...
  g02jd( ...
         vpr, nvpr, gamma, lb, rcomm, icomm, iopt, ropt);

% Print results
fprintf('Number of observations (n)                     = %d\n', n);
fprintf('Number of random factors (nrf)                 = %d\n', nrf);
fprintf('Number of fixed factors (nff)                  = %d\n', nff);
fprintf('Number of subject levels (nlsv)                = %d\n', nlsv);
fprintf('Rank of X (rnkx)                               = %d\n', rnkx);
fprintf('Effective n (effn)                             = %d\n', effn);
fprintf('Number of non-zero variance components (ncov) = %d\n', ncov);

fprintf('\nParameter Estimates\n');

if nrf > 0
  fprintf('\nRandom Effects\n');
end
pb = -999;
pfmt = ' ';
for k = 1:nrf*nlsv
  tb = id(1,k);
  if tb ~= -999
    vid = id(2,k);
    nv = rndm(1,tb);
    ns = rndm(3+nv,tb);
    tfmt = sprintf('%d ', id(3+1:3+ns,k));
    if ( (pb ~= tb) || (strcmp(tfmt, pfmt) == 0) )
      if (k ~= 1)
        fprintf('\n');
      end
      fprintf('  Subject: ');
      for l=1:ns
        fprintf(' Variable %2d (Level %2d)',rndm(3+nv+l,tb), id(3+l,k));
      end
      fprintf('\n');
    end
    if (vid==0)
      % Intercept
      fprintf('    Intercept%18s%10.4f %10.4f\n', ' ', b(k), se(k));
    else
      % variable vid specified in rndm
      aid = rndm(2+vid,tb);
      if (id(3,k)==0)
        fprintf('    Variable %2d%16s%10.4f %10.4f\n', aid, ' ', b(k), se(k));
      else
        fprintf('    Variable %2d (Level %2d)     %10.4f %10.4f\n', ...
                aid, id(3,k), b(k), se(k));
      end
    end
    pfmt = tfmt;
  end
  pb = tb;
end

if nff>0
  fprintf('\nFixed Effects\n');
end
for k = (nrf*nlsv+1):(nrf*nlsv+nff)
  if vid~=-999
    vid = id(2,k);
    if vid==0
      % Intercept
      fprintf('    Intercept%18s%10.4f %10.4f\n', ' ', b(k), se(k));
    else
      % vid'th variable specified in fixed
      aid = fixed(2+vid);
```

```
      if (id(3,k)==0)
        fprintf('    Variable %2d%16s%10.4f %10.4f\n', aid, ' ', b(k), se(k));
      else
        fprintf('    Variable %2d (Level %2d)     %10.4f %10.4f\n', ...
                aid, id(3,k), b(k), se(k));
      end
    end
  end
end

fprintf('\nVariance Components\n');
fprintf('  Estimate     Parameter       Subject\n');
for k = 1:nvpr
  fprintf('%10.5f     ', gamma(k));
  p = 0;
  for tb = 1:nrndm
    nv = rndm(1,tb);
    ns = rndm(3+nv,tb);
    if (rndm(2,tb)==1)
      p = p + 1;
      if (vpr(p)==k)
        fprintf('Intercept       Variables ');
        fprintf('%2d ', rndm(3+nv+1:3+nv+ns, tb));
        fprintf('\n');
      end
    end
    for i = 1:nv
      p = p + 1;
      if (vpr(p)==k)
       fprintf('Variable %2d     Variables %2d ', rndm(2+i,tb));
       fprintf('%2d ', rndm(3+nv+1:3+nv+ns, tb));
      end
    end
  end
  fprintf('\n');
end

fprintf('\nsigma^2          = %15.5f\n', gamma(nvpr+1));
fprintf('-2log likelihood = %15.5f\n', lnlike);
```

### 9.2 Program Results

```
     g02jc example results

Number of observations (n)                    = 90
Number of random factors (nrf)                = 55
Number of fixed factors (nff)                 = 4
Number of subject levels (nlsv)               = 3
Rank of X (rnkx)                              = 4
Effective n (effn)                            = 90
Number of non-zero variance components (ncov) = 7

Parameter Estimates

Random Effects
  Subject:  Variable 10 (Level  1) Variable 11 (Level  1) Variable 12 (Level  1)
    Variable  3 (Level  1)         2.1561     3.7946
    Variable  3 (Level  2)         1.8951     3.9284
    Variable  4 (Level  1)         0.6496     3.1617

  Subject:  Variable 10 (Level  1) Variable 11 (Level  1) Variable 12 (Level  1)
    Variable  4 (Level  3)         0.7390     3.1424

  Subject:  Variable 10 (Level  2) Variable 11 (Level  1) Variable 12 (Level  1)
    Variable  3 (Level  1)         1.4216     3.3773
    Variable  3 (Level  2)        -2.8921     3.3953
    Variable  4 (Level  1)         3.6789     2.3162
    Variable  4 (Level  2)        -1.9742     2.3887
    Variable  4 (Level  3)        -2.2088     2.0697
```

```
Subject:  Variable 10 (Level  1) Variable 11 (Level  2) Variable 12 (Level  1)
  Variable  3 (Level  1)        -2.9659     3.9127
  Variable  3 (Level  2)         2.7951     4.7183
  Variable  4 (Level  1)        -4.7330     2.3094
  Variable  4 (Level  2)         5.5161     2.2330
  Variable  4 (Level  3)        -0.8417     2.3826

Subject:  Variable 10 (Level  2) Variable 11 (Level  2) Variable 12 (Level  1)
  Variable  3 (Level  1)         4.2202     3.6675
  Variable  3 (Level  2)        -4.3883     3.4424
  Variable  4 (Level  1)        -1.1391     3.2187
  Variable  4 (Level  2)         1.0814     3.0654

Subject:  Variable 10 (Level  1) Variable 11 (Level  3) Variable 12 (Level  1)
  Variable  3 (Level  1)         0.3391     4.0647
  Variable  3 (Level  2)         0.1502     3.4787
  Variable  4 (Level  1)        -1.0026     2.4363

Subject:  Variable 10 (Level  1) Variable 11 (Level  3) Variable 12 (Level  1)
  Variable  4 (Level  3)         1.1703     2.6365

Subject:  Variable 10 (Level  2) Variable 11 (Level  3) Variable 12 (Level  1)
  Variable  3 (Level  1)         1.2658     3.4819
  Variable  3 (Level  2)        -1.5356     3.9097

Subject:  Variable 10 (Level  2) Variable 11 (Level  3) Variable 12 (Level  1)
  Variable  4 (Level  2)         0.7992     2.7902
  Variable  4 (Level  3)        -0.8916     2.8763

Subject:  Variable 11 (Level  1) Variable 12 (Level  1)
  Variable  5 (Level  1)        -0.4885     2.8206
  Variable  5 (Level  2)         1.8829     2.7530
  Variable  6 (Level  1)         0.9249     3.7747
  Variable  6 (Level  2)        -2.3568     3.1624
  Variable  6 (Level  3)         4.3117     3.1474

Subject:  Variable 11 (Level  2) Variable 12 (Level  1)
  Variable  5 (Level  1)         1.3898     2.9362
  Variable  5 (Level  2)        -1.5729     2.8909
  Variable  6 (Level  1)         0.2111     3.9967
  Variable  6 (Level  2)        -3.7083     4.2866
  Variable  6 (Level  3)         3.1190     4.7983

Subject:  Variable 11 (Level  3) Variable 12 (Level  1)
  Variable  5 (Level  1)         1.7352     3.1370
  Variable  5 (Level  2)        -1.6165     3.1713
  Variable  6 (Level  1)        -1.1102     3.9374
  Variable  6 (Level  2)         4.4877     3.6980
  Variable  6 (Level  3)        -3.1325     3.1966

Subject:  Variable 12 (Level  1)
  Variable  7                    0.6827     0.5060
  Variable  8 (Level  1)         1.5964     1.3206
  Variable  8 (Level  2)        -0.7533     1.5663
  Variable  8 (Level  3)         0.4035     1.6840
  Variable  8 (Level  4)        -0.8523     1.7518
  Variable  9 (Level  1)         0.5699     1.6236
  Variable  9 (Level  2)         0.0012     1.9111
  Variable  9 (Level  3)        -0.2850     1.9245
  Variable  9 (Level  4)         0.4468     2.0329
  Variable  9 (Level  5)         0.0030     2.1390

Subject:  Variable 10 (Level  1) Variable 11 (Level  1) Variable 12 (Level  2)
  Variable  3 (Level  1)         6.2551     3.3595
  Variable  3 (Level  2)         5.6085     3.4127

Subject:  Variable 10 (Level  1) Variable 11 (Level  1) Variable 12 (Level  2)
  Variable  4 (Level  2)         2.6922     2.7542
  Variable  4 (Level  3)         1.3742     2.8068

Subject:  Variable 10 (Level  2) Variable 11 (Level  1) Variable 12 (Level  2)
```

```
    Variable  3 (Level  1)         1.5647     3.8353
    Variable  3 (Level  2)        -2.7565     3.9041
    Variable  4 (Level  1)        -0.8621     2.8257

  Subject:  Variable 10 (Level  2) Variable 11 (Level  1) Variable 12 (Level  2)
    Variable  4 (Level  3)         0.4536     2.8070

  Subject:  Variable 10 (Level  1) Variable 11 (Level  2) Variable 12 (Level  2)
    Variable  3 (Level  1)       -10.1544     3.3433
    Variable  3 (Level  2)         3.2446     4.1221
    Variable  4 (Level  1)        -2.9419     2.3508
    Variable  4 (Level  2)         0.2510     3.0675
    Variable  4 (Level  3)         0.3224     2.9710

  Subject:  Variable 10 (Level  2) Variable 11 (Level  2) Variable 12 (Level  2)
    Variable  3 (Level  1)        -1.3577     3.1925
    Variable  3 (Level  2)         8.1277     3.9975
    Variable  4 (Level  1)        -0.4290     2.4578
    Variable  4 (Level  2)         2.7495     2.5821

  Subject:  Variable 10 (Level  1) Variable 11 (Level  3) Variable 12 (Level  2)
    Variable  3 (Level  1)         4.8432     4.0069
    Variable  3 (Level  2)         0.0370     3.6006
    Variable  4 (Level  1)         3.0713     2.2706
    Variable  4 (Level  2)        -1.8899     2.4756
    Variable  4 (Level  3)         0.4914     2.2914

  Subject:  Variable 10 (Level  2) Variable 11 (Level  3) Variable 12 (Level  2)
    Variable  3 (Level  1)        -4.4766     3.3355
    Variable  3 (Level  2)        -3.7936     4.0759
    Variable  4 (Level  1)        -0.5459     2.7097
    Variable  4 (Level  2)        -1.5619     2.7412
    Variable  4 (Level  3)        -0.7269     2.9735

  Subject:  Variable 11 (Level  1) Variable 12 (Level  2)
    Variable  5 (Level  1)         4.8653     3.0706
    Variable  5 (Level  2)         0.9011     3.0696
    Variable  6 (Level  1)         6.9277     3.8411
    Variable  6 (Level  2)        -1.3108     3.1667
    Variable  6 (Level  3)         6.2916     3.5327

  Subject:  Variable 11 (Level  2) Variable 12 (Level  2)
    Variable  5 (Level  1)        -0.4047     3.0956
    Variable  5 (Level  2)         0.3291     3.0784
    Variable  6 (Level  1)         6.9096     3.3073
    Variable  6 (Level  2)        -1.0680     3.6213
    Variable  6 (Level  3)        -5.9977     3.7299

  Subject:  Variable 11 (Level  3) Variable 12 (Level  2)
    Variable  5 (Level  1)        -1.0925     3.0994
    Variable  5 (Level  2)        -0.7392     2.9900
    Variable  6 (Level  1)         2.7758     3.8748
    Variable  6 (Level  2)        -6.3526     3.3014
    Variable  6 (Level  3)        -0.2060     3.6481

  Subject:  Variable 12 (Level  2)
    Variable  7                    0.1711     0.5785
    Variable  8 (Level  1)         1.7186     1.9143
    Variable  8 (Level  2)        -0.6768     1.7352
    Variable  8 (Level  3)        -0.0439     1.6395
    Variable  8 (Level  4)         0.1463     1.5358
    Variable  9 (Level  1)         0.9761     2.3930
    Variable  9 (Level  2)         6.5436     1.8193
    Variable  9 (Level  3)        -1.5504     1.8527
    Variable  9 (Level  4)         0.1047     2.0244
    Variable  9 (Level  5)        -3.9386     1.7937

  Subject:  Variable 10 (Level  1) Variable 11 (Level  1) Variable 12 (Level  3)
    Variable  3 (Level  1)        10.6802     3.2596
    Variable  3 (Level  2)        -1.0290     3.7842
    Variable  4 (Level  1)        -2.8612     2.2917
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    Variable  4 (Level  2)         3.9265     2.8934
    Variable  4 (Level  3)         2.2427     2.3737

  Subject:  Variable 10 (Level  2) Variable 11 (Level  1) Variable 12 (Level  3)
    Variable  3 (Level  1)        -6.2076     3.3642
    Variable  3 (Level  2)        -8.7670     3.8463
    Variable  4 (Level  1)        -2.9251     2.4657

  Subject:  Variable 10 (Level  2) Variable 11 (Level  1) Variable 12 (Level  3)
    Variable  4 (Level  3)        -2.2077     2.3743

  Subject:  Variable 10 (Level  1) Variable 11 (Level  2) Variable 12 (Level  3)
    Variable  3 (Level  1)        -3.3334     3.4665
    Variable  3 (Level  2)        -0.3111     3.2650
    Variable  4 (Level  1)         1.5131     2.4890
    Variable  4 (Level  2)        -3.0345     3.0562
    Variable  4 (Level  3)         0.2722     2.8300

  Subject:  Variable 10 (Level  2) Variable 11 (Level  2) Variable 12 (Level  3)
    Variable  3 (Level  1)         6.5905     4.0386
    Variable  3 (Level  2)        -5.3168     3.4549
    Variable  4 (Level  1)        -3.5280     2.9663
    Variable  4 (Level  2)         1.7056     2.9293
    Variable  4 (Level  3)         2.2590     3.1780

  Subject:  Variable 10 (Level  1) Variable 11 (Level  3) Variable 12 (Level  3)
    Variable  3 (Level  1)         8.1889     4.1429
    Variable  3 (Level  2)        -1.5388     3.3333
    Variable  4 (Level  1)         3.4338     2.6376

  Subject:  Variable 10 (Level  1) Variable 11 (Level  3) Variable 12 (Level  3)
    Variable  4 (Level  3)        -1.1544     2.9885

  Subject:  Variable 10 (Level  2) Variable 11 (Level  3) Variable 12 (Level  3)
    Variable  3 (Level  1)        -4.4243     4.0049
    Variable  3 (Level  2)        -4.1349     3.1248
    Variable  4 (Level  1)         1.0460     2.6550
    Variable  4 (Level  2)        -4.4844     2.2843
    Variable  4 (Level  3)         0.5046     2.6926

  Subject:  Variable 11 (Level  1) Variable 12 (Level  3)
    Variable  5 (Level  1)         5.3030     3.0278
    Variable  5 (Level  2)        -8.1794     3.1335
    Variable  6 (Level  1)        -0.8188     3.7810
    Variable  6 (Level  2)        -2.5078     3.1514
    Variable  6 (Level  3)        -2.6138     3.4600

  Subject:  Variable 11 (Level  2) Variable 12 (Level  3)
    Variable  5 (Level  1)         4.3331     3.1489
    Variable  5 (Level  2)        -5.6142     3.1649
    Variable  6 (Level  1)        -5.8804     3.1770
    Variable  6 (Level  2)         5.4265     3.3006
    Variable  6 (Level  3)        -2.1917     3.2156

  Subject:  Variable 11 (Level  3) Variable 12 (Level  3)
    Variable  5 (Level  1)         0.4305     2.9144
    Variable  5 (Level  2)        -1.4620     3.0119
    Variable  6 (Level  1)        14.3595     3.9254
    Variable  6 (Level  2)        -5.2399     3.3099
    Variable  6 (Level  3)       -11.2498     3.2212

  Subject:  Variable 12 (Level  3)
    Variable  7                   -0.3839     0.6755
    Variable  8 (Level  1)         2.7549     1.6017
    Variable  8 (Level  2)         0.4377     1.8826
    Variable  8 (Level  3)        -0.2261     1.9909
    Variable  8 (Level  4)        -4.5051     1.5398
    Variable  9 (Level  1)        -4.7091     2.1458
    Variable  9 (Level  2)         3.7940     1.9872
    Variable  9 (Level  3)        -1.7994     1.8614
    Variable  9 (Level  4)         0.4480     1.9016
```

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    Variable  9 (Level  5)        -0.6047     2.4729

Fixed Effects
    Intercept                      1.6433     2.4596
    Variable  1 (Level  2)        -1.6224     0.8549
    Variable  2 (Level  2)        -2.4817     1.1414
    Variable  2 (Level  3)         0.4624     1.2133

Variance Components
  Estimate     Parameter       Subject
  36.32491     Variable  3     Variables 10 11 12
  12.45090     Variable  4     Variables 10 11 12
  19.62767     Variable  5     Variables 11 12
  40.53480     Variable  6     Variables 11 12
   0.56320     Variable  7     Variables 12
   5.81968     Variable  8     Variables 12
  10.86069     Variable  9     Variables 12

sigma^2          =         0.00239
-2log likelihood =       608.19449
```