NAG Toolbox

nag_matop_complex_gen_matrix_cond_pow (f01ke)

1 Purpose

nag_matop_complex_gen_matrix_cond_pow (f01ke) computes an estimate of the relative condition number κ_{A^p} of the pth power (where p is real) of a complex n by n matrix A, in the 1-norm. The principal matrix power A^p is also returned.

2 Syntax

```
[a, condpa, ifail] = nag_matop_complex_gen_matrix_cond_pow(a, p, 'n', n)
[a, condpa, ifail] = f01ke(a, p, 'n', n)
```

3 Description

For a matrix A with no eigenvalues on the closed negative real line, A^p ($p \in \mathbb{R}$) can be defined as

$$A^p = \exp(p\log(A))$$

where $\log(A)$ is the principal logarithm of A (the unique logarithm whose spectrum lies in the strip $\{z : -\pi < \mathrm{Im}(z) < \pi\}$).

The Fréchet derivative of the matrix pth power of A is the unique linear mapping $E \mapsto L(A,E)$ such that for any matrix E

$$(A+E)^p - A^p - L(A,E) = o(\|E\|).$$

The derivative describes the first-order effect of perturbations in A on the matrix power A^p.

The relative condition number of the matrix pth power can be defined by

$$\kappa_{A^p} = \frac{\|L(A)\|\|A\|}{\|A^p\|},$$

where $\|L(A)\|$ is the norm of the Fréchet derivative of the matrix power at A.

nag_matop_complex_gen_matrix_cond_pow (f01ke) uses the algorithms of Higham and Lin (2011) and Higham and Lin (2013) to compute κ_{A^p} and A^p. The real number p is expressed as $p = q + r$ where $q \in (-1,1)$ and $r \in \mathbb{Z}$. Then $A^p = A^q A^r$. The integer power A^r is found using a combination of binary powering and, if necessary, matrix inversion. The fractional power A^q is computed using a Schur decomposition, a Padé approximant and the scaling and squaring method.

To obtain the estimate of κ_{A^p}, nag_matop_complex_gen_matrix_cond_pow (f01ke) first estimates $\|L(A)\|$ by computing an estimate γ of a quantity $K \in \left[n^{-1}\|L(A)\|_1, n\|L(A)\|_1\right]$, such that $\gamma \leq K$. This requires multiple Fréchet derivatives to be computed. Fréchet derivatives of A^q are obtained by differentiating the Padé approximant. Fréchet derivatives of A^p are then computed using a combination of the chain rule and the product rule for Fréchet derivatives.

If A is nonsingular but has negative real eigenvalues nag_matop_complex_gen_matrix_cond_pow (f01ke) will return a non-principal matrix pth power and its condition number.

4 References

Higham N J (2008) *Functions of Matrices: Theory and Computation* SIAM, Philadelphia, PA, USA

Higham N J and Lin L (2011) A Schur–Padé algorithm for fractional powers of a matrix *SIAM J. Matrix Anal. Appl.* **32(3)** 1056–1078

Higham N J and Lin L (2013) An improved Schur–Padé algorithm for fractional powers of a matrix and their Fréchet derivatives *MIMS Eprint 2013.1* Manchester Institute for Mathematical Sciences, School of Mathematics, University of Manchester http://eprints.ma.man.ac.uk/

5 Parameters

5.1 Compulsory Input Parameters

1: **a**$(lda,:)$ – COMPLEX (KIND=nag_wp) array

The first dimension of the array **a** must be at least **n**.

The second dimension of the array **a** must be at least **n**.

The n by n matrix A.

2: **p** – REAL (KIND=nag_wp)

The required power of A.

5.2 Optional Input Parameters

1: **n** – INTEGER

Default: the first dimension of the array **a** and the second dimension of the array **a**. (An error is raised if these dimensions are not equal.)

n, the order of the matrix A.

Constraint: $\mathbf{n} \geq 0$.

5.3 Output Parameters

1: **a**$(lda,:)$ – COMPLEX (KIND=nag_wp) array

The first dimension of the array **a** will be **n**.

The second dimension of the array **a** will be **n**.

The n by n principal matrix pth power, A^p, unless $\mathbf{ifail} = 1$, in which case a non-principal pth power is returned.

2: **condpa** – REAL (KIND=nag_wp)

If $\mathbf{ifail} = 0$ or 3, an estimate of the relative condition number of the matrix pth power, κ_{A^p}. Alternatively, if $\mathbf{ifail} = 4$, the absolute condition number of the matrix pth power.

3: **ifail** – INTEGER

$\mathbf{ifail} = 0$ unless the function detects an error (see Section 5).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

$\mathbf{ifail} = 1$

A has eigenvalues on the negative real line. The principal pth power is not defined in this case, so a non-principal power was returned.

ifail $= 2$

A is singular so the pth power cannot be computed.

ifail $= 3$

A^p has been computed using an IEEE double precision Padé approximant, although the arithmetic precision is higher than IEEE double precision.

ifail $= 4$

The relative condition number is infinite. The absolute condition number was returned instead.

ifail $= 5$

An unexpected internal error occurred. This failure should not occur and suggests that the function has been called incorrectly.

ifail $= -1$

Constraint: **n** ≥ 0.

ifail $= -3$

Constraint: $lda \geq$ **n**.

ifail $= -99$

An unexpected error has been triggered by this routine. Please contact NAG.

ifail $= -399$

Your licence key may have expired or may not have been installed correctly.

ifail $= -999$

Dynamic memory allocation failed.

7 Accuracy

nag_matop_complex_gen_matrix_cond_pow (f01ke) uses the norm estimation function nag_linsys_complex_gen_norm_rcomm (f04zd) to produce an estimate γ of a quantity $K \in \left[n^{-1}\|L(A)\|_1, n\|L(A)\|_1\right]$, such that $\gamma \leq K$. For further details on the accuracy of norm estimation, see the documentation for nag_linsys_complex_gen_norm_rcomm (f04zd).

For a normal matrix A (for which $A^{\mathrm{H}}A = AA^{\mathrm{H}}$), the Schur decomposition is diagonal and the computation of the fractional part of the matrix power reduces to evaluating powers of the eigenvalues of A and then constructing A^p using the Schur vectors. This should give a very accurate result. In general, however, no error bounds are available for the algorithm. See Higham and Lin (2011) and Higham and Lin (2013) for details and further discussion.

8 Further Comments

The amount of complex allocatable memory required by the algorithm is typically of the order $10 \times n^2$.

The cost of the algorithm is $O(n^3)$ floating-point operations; see Higham and Lin (2013).

If the matrix pth power alone is required, without an estimate of the condition number, then nag_matop_complex_gen_matrix_pow (f01fq) should be used. If the Fréchet derivative of the matrix power is required then nag_matop_complex_gen_matrix_frcht_pow (f01kf) should be used. The real analogue of this function is nag_matop_real_gen_matrix_cond_pow (f01je).

9 Example

This example estimates the relative condition number of the matrix power A^p, where $p = 0.4$ and

$$A = \begin{pmatrix} 1+2i & 3 & 2 & 1+3i \\ 1+i & 1 & 1 & 2+i \\ 1 & 2 & 1 & 2i \\ 3 & i & 2+i & 1 \end{pmatrix}.$$

9.1 Program Text

```
function f01ke_example

fprintf('f01ke example results\n\n');

% Principal power p of matrix A

a = [ 1+2i  3      2     1+3i;
      1+ i  1      1     2+ i;
      1     2      1       2i;
      3        i   2+i   1];

p = 0.4;

[pa, condpa, ifail] = f01ke(a,p);

disp('A^p:');
disp(pa);

fprintf('Estimated condition number is: %6.2f\n', condpa)
```

9.2 Program Results

```
f01ke example results

A^p:
   0.9742 + 0.5211i   0.8977 - 0.1170i   0.6389 - 0.3900i   0.0975 + 0.6205i
   0.1586 + 0.2763i   1.0176 - 0.0250i   0.0623 - 0.3471i   0.6431 + 0.2560i
   0.2589 - 0.5817i   0.5633 + 0.3969i   1.1470 + 0.4042i  -0.3771 + 0.3113i
   0.8713 - 0.0270i  -0.5734 + 0.0868i   0.2816 + 0.3739i   1.3568 - 0.2709i

Estimated condition number is:   6.86
```