NAG Toolbox

nag_inteq_fredholm2_split (d05aa)

1 Purpose

nag_inteq_fredholm2_split (d05aa) solves a linear, nonsingular Fredholm equation of the second kind with a split kernel.

2 Syntax

```
[f, c, ifail] = nag_inteq_fredholm2_split(lambda, a, b, k1, k2, g, n, ind)
[f, c, ifail] = d05aa(lambda, a, b, k1, k2, g, n, ind)
```

3 Description

nag_inteq_fredholm2_split (d05aa) solves an integral equation of the form

$$f(x) - \lambda \int_a^b k(x,s) f(s)\, ds = g(x)$$

for $a \leq x \leq b$, when the kernel k is defined in two parts: $k = k_1$ for $a \leq s \leq x$ and $k = k_2$ for $x < s \leq b$. The method used is that of El–Gendi (1969) for which, it is important to note, each of the functions k_1 and k_2 must be defined, smooth and nonsingular, for all x and s in the interval $[a, b]$.

An approximation to the solution $f(x)$ is found in the form of an n term Chebyshev series $\sum_{i=1}^{n}{}' c_i T_i(x)$, where $'$ indicates that the first term is halved in the sum. The coefficients c_i, for $i = 1, 2, \ldots, n$, of this series are determined directly from approximate values f_i, for $i = 1, 2, \ldots, n$, of the function $f(x)$ at the first n of a set of $m + 1$ Chebyshev points:

$$x_i = \tfrac{1}{2}(a + b + (b - a)\cos[(i - 1)\pi/m]), \quad i = 1, 2, \ldots, m + 1.$$

The values f_i are obtained by solving simultaneous linear algebraic equations formed by applying a quadrature formula (equivalent to the scheme of Clenshaw and Curtis (1960)) to the integral equation at the above points.

In general $m = n - 1$. However, if the kernel k is centro-symmetric in the interval $[a, b]$, i.e., if $k(x, s) = k(a + b - x, a + b - s)$, then the function is designed to take advantage of this fact in the formation and solution of the algebraic equations. In this case, symmetry in the function $g(x)$ implies symmetry in the function $f(x)$. In particular, if $g(x)$ is even about the mid-point of the range of integration, then so also is $f(x)$, which may be approximated by an even Chebyshev series with $m = 2n - 1$. Similarly, if $g(x)$ is odd about the mid-point then $f(x)$ may be approximated by an odd series with $m = 2n$.

4 References

Clenshaw C W and Curtis A R (1960) A method for numerical integration on an automatic computer *Numer. Math.* **2** 197–205

El–Gendi S E (1969) Chebyshev solution of differential, integral and integro-differential equations *Comput. J.* **12** 282–287

5 Parameters

5.1 Compulsory Input Parameters

1: **lambda** – REAL (KIND=nag_wp)

The value of the parameter λ of the integral equation.

2: **a** – REAL (KIND=nag_wp)

a, the lower limit of integration.

3: **b** – REAL (KIND=nag_wp)

b, the upper limit of integration.

Constraint: **b** $>$ **a**.

4: **k1** – REAL (KIND=nag_wp) FUNCTION, supplied by the user.

k1 must evaluate the kernel $k(x,s) = k_1(x,s)$ of the integral equation for $a \leq s \leq x$.

```
[result] = k1(x, s)
```

Input Parameters

1: **x** – REAL (KIND=nag_wp)
2: **s** – REAL (KIND=nag_wp)

The values of x and s at which $k_1(x,s)$ is to be evaluated.

Output Parameters

1: **result**

The value of the kernel $k(x,s) = k_1(x,s)$ evaluated at **x** and **s**.

5: **k2** – REAL (KIND=nag_wp) FUNCTION, supplied by the user.

k2 must evaluate the kernel $k(x,s) = k_2(x,s)$ of the integral equation for $x < s \leq b$.

```
[result] = k2(x, s)
```

Input Parameters

1: **x** – REAL (KIND=nag_wp)
2: **s** – REAL (KIND=nag_wp)

The values of x and s at which $k_2(x,s)$ is to be evaluated.

Output Parameters

1: **result**

The value of the kernel $k(x,s) = k_2(x,s)$ evaluated at **x** and **s**.

Note that the functions k_1 and k_2 must be defined, smooth and nonsingular for all x and s in the interval $[a,b]$.

6: **g** – REAL (KIND=nag_wp) FUNCTION, supplied by the user.

g must evaluate the function $g(x)$ for $a \leq x \leq b$.

```
[result] = g(x)
```

Input Parameters

1: **x** – REAL (KIND=nag_wp)

The values of x at which $g(x)$ is to be evaluated.

Output Parameters

1: **result**

The value of $g(x)$ evaluated at **x**.

7: **n** – INTEGER

The number of terms in the Chebyshev series required to approximate $f(x)$.

Constraint: $\mathbf{n} \geq 1$.

8: **ind** – INTEGER

Determines the forms of the kernel, $k(x,s)$, and the function $g(x)$.

$\mathbf{ind} = 0$
$k(x,s)$ is not centro-symmetric (or no account is to be taken of centro-symmetry).

$\mathbf{ind} = 1$
$k(x,s)$ is centro-symmetric and $g(x)$ is odd.

$\mathbf{ind} = 2$
$k(x,s)$ is centro-symmetric and $g(x)$ is even.

$\mathbf{ind} = 3$
$k(x,s)$ is centro-symmetric but $g(x)$ is neither odd nor even.

Constraint: $\mathbf{ind} = 0$, 1, 2 or 3.

5.2 Optional Input Parameters

None.

5.3 Output Parameters

1: **f**(**n**) – REAL (KIND=nag_wp) array

The approximate values f_i, for $i = 1, 2, \ldots, \mathbf{n}$, of $f(x)$ evaluated at the first **n** of $m + 1$ Chebyshev points x_i, (see Section 3).

If $\mathbf{ind} = 0$ or 3, $m = \mathbf{n} - 1$.

If $\mathbf{ind} = 1$, $m = 2 \times \mathbf{n}$.

If $\mathbf{ind} = 2$, $m = 2 \times \mathbf{n} - 1$.

2: **c**(**n**) – REAL (KIND=nag_wp) array

The coefficients c_i, for $i = 1, 2, \ldots, \mathbf{n}$, of the Chebyshev series approximation to $f(x)$.

If $\mathbf{ind} = 1$ this series contains polynomials of odd order only and if $\mathbf{ind} = 2$ the series contains even order polynomials only.

3: **ifail** – INTEGER

$\mathbf{ifail} = 0$ unless the function detects an error (see Section 5).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

ifail $= 1$

On entry, **a** $\geq$ **b** or **n** < 1.

ifail $= 2$

A failure has occurred due to proximity to an eigenvalue. In general, if **lambda** is near an eigenvalue of the integral equation, the corresponding matrix will be nearly singular. In the special case, $m = 1$, the matrix reduces to a zero-valued number.

ifail $= -99$

An unexpected error has been triggered by this routine. Please contact NAG.

ifail $= -399$

Your licence key may have expired or may not have been installed correctly.

ifail $= -999$

Dynamic memory allocation failed.

7 Accuracy

No explicit error estimate is provided by the function but it is usually possible to obtain a good indication of the accuracy of the solution either

(i) by examining the size of the later Chebyshev coefficients c_i, or

(ii) by comparing the coefficients c_i or the function values f_i for two or more values of **n**.

8 Further Comments

The time taken by nag_inteq_fredholm2_split (d05aa) increases with **n**.

This function may be used to solve an equation with a continuous kernel by defining **k1** and **k2** to be identical.

This function may also be used to solve a Volterra equation by defining **k2** (or **k1**) to be identically zero.

9 Example

This example solves the equation

$$f(x) - \int_0^1 k(x,s) f(s)\, ds = \left(1 - \frac{1}{\pi^2}\right) \sin(\pi x)$$

where

$$k(x,s) = \begin{cases} s(1-x) & \text{for } 0 \leq s \leq x, \\ x(1-s) & \text{for } x < s \leq 1. \end{cases}$$

Five terms of the Chebyshev series are sought, taking advantage of the centro-symmetry of the $k(x,s)$ and even nature of $g(x)$ about the mid-point of the range $[0,1]$.

The approximate solution at the point $x = 0.1$ is calculated by calling nag_sum_chebyshev (c06dc).

9.1 Program Text

```
function d05aa_example

fprintf('d05aa example results\n\n');

lambda = 1;
a = 0;
b = 1;
g = @(x) sin(pi*x)*(1-1/(pi*pi));
k1 = @(x, s) s*(1-x);
k2 = @(x, s) x*(1-s);
n = nag_int(5);
ind = nag_int(2);
[f, c, ifail] = d05aa(lambda, a, b, k1, k2, g, n, ind);

xval = 0.1;

% evaluate Chebyshev series at xval
s = nag_int(2);
[res, ifail] = c06dc(xval, a, b, c, s);
fprintf('Kernel is centro-symmetric and G is even so the solution is even\n')
fprintf('\nChebyshev coefficients:\n');
fprintf('%14.4f',c);
fprintf('\n\n x = %5.2f    Ans = %7.4f\n',xval,res);
```

9.2 Program Results

```
d05aa example results

Kernel is centro-symmetric and G is even so the solution is even

Chebyshev coefficients:
        0.9440       -0.4994        0.0280       -0.0006        0.0000

 x =  0.10    Ans =  0.3090
```