NAG Toolbox

nag_pde_1d_blackscholes_closed (d03nd)

1 Purpose

nag_pde_1d_blackscholes_closed (d03nd) computes an analytic solution to the Black–Scholes equation for a certain set of option types.

2 Syntax

```
[f, theta, delta, gamma, lambda, rho, ifail] = nag_pde_1d_blackscholes_closed
(kopt, x, s, t, tmat, tdpar, r, q, sigma)
```

```
[f, theta, delta, gamma, lambda, rho, ifail] = d03nd(kopt, x, s, t, tmat, tdpar,
r, q, sigma)
```

3 Description

nag_pde_1d_blackscholes_closed (d03nd) computes an analytic solution to the Black–Scholes equation (see Hull (1989) and Wilmott *et al.* (1995))

$$\frac{\partial f}{\partial t} + (r - q)S\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} = rf \tag{1}$$

$$S_{\text{min}} < S < S_{\text{max}}, \quad t_{\text{min}} < t < t_{\text{max}}, \tag{2}$$

for the value f of a European put or call option, or an American call option with zero dividend q. In equation (1) t is time, S is the stock price, X is the exercise price, r is the risk free interest rate, q is the continuous dividend, and σ is the stock volatility. The parameter r, q and σ may be either constant, or functions of time. In the latter case their average instantaneous values over the remaining life of the option should be provided to nag_pde_1d_blackscholes_closed (d03nd). An auxiliary function nag_pde_1d_blackscholes_means (d03ne) is available to compute such averages from values at a set of discrete times. Equation (1) is subject to different boundary conditions depending on the type of option. For a call option the boundary condition is

$$f(S, t = t_{\text{mat}}) = \max(0, S - X)$$

where t_{mat} is the maturity time of the option. For a put option the equation (1) is subject to

$$f(S, t = t_{\text{mat}}) = \max(0, X - S).$$

nag_pde_1d_blackscholes_closed (d03nd) also returns values of the Greeks

$$\Theta = \frac{\partial f}{\partial t}, \quad \Delta = \frac{\partial f}{\partial x}, \quad \Gamma = \frac{\partial^2 f}{\partial x^2}, \quad \Lambda = \frac{\partial f}{\partial \sigma}, \quad \rho = \frac{\partial f}{\partial r}.$$

nag_specfun_opt_bsm_greeks (s30ab) also computes the European option price given by the Black–Scholes–Merton formula together with a more comprehensive set of sensitivities (Greeks).

Further details of the analytic solution returned are given in Section 9.1.

4 References

Hull J (1989) *Options, Futures and Other Derivative Securities* Prentice–Hall

Wilmott P, Howison S and Dewynne J (1995) *The Mathematics of Financial Derivatives* Cambridge University Press

5 Parameters

5.1 Compulsory Input Parameters

1: **kopt** – INTEGER

Specifies the kind of option to be valued:

kopt $= 1$
A European call option.

kopt $= 2$
An American call option.

kopt $= 3$
A European put option.

Constraints:

kopt $= 1$, 2 or 3;
if $q \neq 0$, **kopt** $\neq 2$.

2: **x** – REAL (KIND=nag_wp)

The exercise price X.

Constraint: **x** ≥ 0.0.

3: **s** – REAL (KIND=nag_wp)

The stock price at which the option value and the Greeks should be evaluated.

Constraint: **s** ≥ 0.0.

4: **t** – REAL (KIND=nag_wp)

The time at which the option value and the Greeks should be evaluated.

Constraint: **t** ≥ 0.0.

5: **tmat** – REAL (KIND=nag_wp)

The maturity time of the option.

Constraint: **tmat** $\geq$ **t**.

6: **tdpar(3)** – LOGICAL array

Specifies whether or not various arguments are time-dependent. More precisely, r is time-dependent if **tdpar**$(1) = \mathit{true}$ and constant otherwise. Similarly, **tdpar**(2) specifies whether q is time-dependent and **tdpar**(3) specifies whether σ is time-dependent.

7: **r**(:) – REAL (KIND=nag_wp) array

The dimension of the array **r** must be at least 3 if **tdpar**$(1) = \mathit{true}$, and at least 1 otherwise

If **tdpar**$(1) = \mathit{false}$ then **r**(1) must contain the constant value of r. The remaining elements need not be set.

If **tdpar**$(1) = \mathit{true}$ then **r**(1) must contain the value of r at time **t** and **r**(2) must contain its average instantaneous value over the remaining life of the option:

$$\hat{r} = \int_{\mathbf{t}}^{\mathbf{tmat}} r(\zeta)\, d\zeta.$$

The auxiliary function nag_pde_1d_blackscholes_means (d03ne) may be used to construct **r** from a set of values of r at discrete times.

8: **q**(:) – REAL (KIND=nag_wp) array

The dimension of the array **q** must be at least 3 if **tdpar**$(2) = \mathit{true}$, and at least 1 otherwise

If **tdpar**$(2) = \mathit{false}$ then **q**(1) must contain the constant value of q. The remaining elements need not be set.

If **tdpar**$(2) = \mathit{true}$ then **q**(1) must contain the constant value of q and **q**(2) must contain its average instantaneous value over the remaining life of the option:

$$\hat{q} = \int_{\mathbf{t}}^{\mathbf{tmat}} q(\zeta)\, d\zeta.$$

The auxiliary function nag_pde_1d_blackscholes_means (d03ne) may be used to construct **q** from a set of values of q at discrete times.

9: **sigma**(:) – REAL (KIND=nag_wp) array

The dimension of the array **sigma** must be at least 3 if **tdpar**$(3) = \mathit{true}$, and at least 1 otherwise

If **tdpar**$(3) = \mathit{false}$ then **sigma**(1) must contain the constant value of σ. The remaining elements need not be set.

If **tdpar**$(3) = \mathit{true}$ then **sigma**(1) must contain the value of σ at time **t**, **sigma**(2) the average instantaneous value $\hat{\sigma}$, and **sigma**(3) the second-order average $\bar{\sigma}$, where:

$$\hat{\sigma} = \int_{\mathbf{t}}^{\mathbf{tmat}} \sigma(\zeta)\, d\zeta,$$

$$\bar{\sigma} = \left(\int_{\mathbf{t}}^{\mathbf{tmat}} \sigma^2(\zeta)\, d\zeta \right)^{1/2}.$$

The auxiliary function nag_pde_1d_blackscholes_means (d03ne) may be used to compute **sigma** from a set of values at discrete times.

Constraints:

if **tdpar**$(3) = \mathit{false}$, **sigma**$(1) > 0.0$;
if **tdpar**$(3) = \mathit{true}$, **sigma**$(i) > 0.0$, for $i = 1, 2, 3$.

5.2 Optional Input Parameters

None.

5.3 Output Parameters

1: **f** – REAL (KIND=nag_wp)

The value f of the option at the stock price **s** and time **t**.

2: **theta** – REAL (KIND=nag_wp)
3: **delta** – REAL (KIND=nag_wp)
4: **gamma** – REAL (KIND=nag_wp)
5: **lambda** – REAL (KIND=nag_wp)
6: **rho** – REAL (KIND=nag_wp)

The values of various Greeks at the stock price **s** and time **t**, as follows:

$$\mathbf{theta} = \Theta = \frac{\partial f}{\partial t}, \qquad \mathbf{delta} = \Delta = \frac{\partial f}{\partial \mathbf{s}}, \quad \mathbf{gamma} = \Gamma = \frac{\partial^2 f}{\partial \mathbf{s}^2},$$

$$\mathbf{lambda} = \Lambda = \frac{\partial f}{\partial \sigma}, \quad \mathbf{rho} = \rho = \frac{\partial f}{\partial r}.$$

7: **ifail** – INTEGER

ifail $= 0$ unless the function detects an error (see Section 5).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

ifail $= 1$

On entry, **kopt** < 1,
or **kopt** > 3,
or **kopt** $= 2$ when $q \neq 0$,
or **x** < 0.0,
or **s** < 0.0,
or **t** < 0.0,
or **tmat** $<$ **t**,
or **sigma**$(1) \leq 0.0$, with **tdpar**$(3) = \mathit{false}$,
or **sigma**$(i) \leq 0.0$, with **tdpar**$(3) = \mathit{true}$, for some $i = 1$, 2 or 3.

ifail $= -99$

An unexpected error has been triggered by this routine. Please contact NAG.

ifail $= -399$

Your licence key may have expired or may not have been installed correctly.

ifail $= -999$

Dynamic memory allocation failed.

7 Accuracy

Given accurate values of **r**, **q** and **sigma** no further approximations are made in the evaluation of the Black–Scholes analytic formulae, and the results should therefore be within machine accuracy. The values of **r**, **q** and **sigma** returned from nag_pde_1d_blackscholes_means (d03ne) are exact for polynomials of degree up to 3.

8 Further Comments

8.1 Algorithmic Details

The Black–Scholes analytic formulae are used to compute the solution. For a European call option these are as follows:

$$f = Se^{-\hat{q}(T-t)}N(d_1) - Xe^{-\hat{r}(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/X) + \left(\hat{r} - \hat{q} + \bar{\sigma}^2/2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}},$$

$$d_2 = \frac{\log(S/X) + \left(\hat{r} - \hat{q} - \bar{\sigma}^2/2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}} = d_1 - \bar{\sigma}\sqrt{T-t},$$

$N(x)$ is the cumulative Normal distribution function and $N'(x)$ is its derivative

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\zeta^2/2} d\zeta,$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The functions $\hat{q}$, $\hat{r}$, $\hat{\sigma}$ and $\bar{\sigma}$ are average values of q, r and σ over the time to maturity:

$$\hat{q} = \frac{1}{T-t} \int_t^T q(\zeta) d\zeta,$$

$$\hat{r} = \frac{1}{T-t} \int_t^T r(\zeta) d\zeta,$$

$$\hat{\sigma} = \frac{1}{T-t} \int_t^T \sigma(\zeta) d\zeta,$$

$$\bar{\sigma} = \left(\frac{1}{T-t} \int_t^T \sigma^2(\zeta) d\zeta \right)^{1/2}.$$

The Greeks are then calculated as follows:

$$\Delta = \frac{\partial f}{\partial S} = e^{-\hat{q}(T-t)} N(d_1) + \frac{S e^{-\hat{q}(T-t)} N'(d_1) - X e^{-\hat{r}(T-t)} N'(d_2)}{\bar{\sigma} S \sqrt{T-t}},$$

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{S e^{-\hat{q}(T-t)} N'(d_1) + X e^{-\hat{r}(T-t)} N'(d_2)}{\bar{\sigma} S^2 \sqrt{T-t}} + \frac{S e^{-\hat{q}(T-t)} N'(d_1) - X e^{-\hat{r}(T-t)} N'(d_2)}{\bar{\sigma}^2 S^2 (T-t)},$$

$$\Theta = \frac{\partial f}{\partial t} = rf + (q-r) S \Delta - \frac{\sigma^2 S^2}{2} \Gamma,$$

$$\Lambda = \frac{\partial f}{\partial \sigma} = \left(\frac{X d_1 e^{-\hat{r}(T-t)} N'(d_2) - S d_2 e^{-\hat{q}(T-t)} N'(d_1)}{\bar{\sigma}^2} \right) \hat{\sigma},$$

$$\rho = \frac{\partial f}{\partial r} = X(T-t) e^{-\hat{r}(T-t)} N(d_2) + \frac{\left(S e^{-\hat{q}(T-t)} N'(d_1) - X e^{-\hat{r}(T-t)} N'(d_2) \right) \sqrt{T-t}}{\bar{\sigma}}.$$

Note: that Θ is obtained from substitution of other Greeks in the Black–Scholes partial differential equation, rather than differentiation of f. The values of q, r and σ appearing in its definition are the instantaneous values, not the averages. Note also that both the first-order average $\hat{\sigma}$ and the second-order average $\bar{\sigma}$ appear in the expression for Λ. This results from the fact that Λ is the derivative of f with respect to σ, not $\hat{\sigma}$.

For a European put option the equivalent equations are:

$$f = Xe^{-\hat{r}(T-t)}N(-d_2) - Se^{-\hat{q}(T-t)}N(-d_1),$$

$$\Delta = \frac{\partial f}{\partial S} = -e^{-\hat{q}(T-t)}N(-d_1) + \frac{Se^{-\hat{q}(T-t)}N'(-d_1) - Xe^{-\hat{r}(T-t)}N'(-d_2)}{\bar{\sigma}S\sqrt{T-t}},$$

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{Xe^{-\hat{r}(T-t)}N'(-d_2) + Se^{-\hat{q}(T-t)}N'(-d_1)}{\bar{\sigma}S^2\sqrt{T-t}} + \frac{Xe^{-\hat{r}(T-t)}N''(-d_2) - Se^{-\hat{q}(T-t)}N''(-d_1)}{\bar{\sigma}^2 S^2(T-t)},$$

$$\Theta = \frac{\partial f}{\partial t} = rf + (q-r)S\Delta - \frac{\sigma^2 S^2}{2}\Gamma,$$

$$\Lambda = \frac{\partial f}{\partial \sigma} = \left(\frac{Xd_1e^{-\hat{r}(T-t)}N'(-d_2) - Sd_2e^{-\hat{q}(T-t)}N'(-d_1)}{\bar{\sigma}^2}\right)\hat{\sigma},$$

$$\rho = \frac{\partial f}{\partial r} = -X(T-t)e^{-\hat{r}(T-t)}N(-d_2) + \frac{\left(Se^{-\hat{q}(T-t)}N'(-d_1) - Xe^{-\hat{r}(T-t)}N'(-d_2)\right)\sqrt{T-t}}{\hat{\sigma}}.$$

The analytic solution for an American call option with $q = 0$ is identical to that for a European call, since early exercise is never optimal in this case. For all other cases no analytic solution is known.

9 Example

This example solves the Black–Scholes equation for valuation of a 5-month American call option on a non-dividend-paying stock with an exercise price of \$50. The risk-free interest rate is 10% per annum, and the stock volatility is 40% per annum.

The option is valued at a range of times and stock prices.

9.1 Program Text

```
     function d03nd_example

fprintf('d03nd example results\n\n');

% American 5-month call option, exercise price 50

kopt = nag_int(2);
x = 50;
ns    = 21;  nt    = 4;
s_beg = 0;   t_beg = 0;
s_end = 100; t_end = 0.125;
tmat  = 0.4166667;
tdpar = [false; false; false];
r     = [0.1];
q     = [0];
sigma = [0.4];

% Discretize s and t
ds = (s_end-s_beg)/(ns-1);
dt = (t_end-t_beg)/(nt-1);
s = [s_beg:ds:s_end];
t = [t_beg:dt:t_end];

f = zeros(ns,nt);
theta = f; delta = f; gamma = f; lambda = f; rho = f;

% Loop over times and prices
```

```
for j = 1:nt
  for i = 1:ns
    [f(i,j),theta(i,j),delta(i,j),gamma(i,j),lambda(i,j),rho(i,j),ifail] = ...
    d03nd( ...
           kopt, x, s(i), t(j), tmat, tdpar, r, q, sigma);
  end
end

% Tabulate option values only
print_greek(ns,nt,tmat,s,t,'Option Values',f);
% print_greek(ns,nt,tmat,s,t,'Theta',theta);
% print_greek(ns,nt,tmat,s,t,'Delta',delta);
% print_greek(ns,nt,tmat,s,t,'Gamma',gamma);
% print_greek(ns,nt,tmat,s,t,'Lambda',lambda);
% print_greek(ns,nt,tmat,s,t,'Rho',rho);

% plot initial and final option values and greeks
fig1 = figure;
plot(s,f(:,1),s,theta(:,1),s,delta(:,1),s,gamma(:,1),s,lambda(:,1),s,rho(:,1));
legend('value','theta','delta','gamma','lambda','rho','Location','NorthWest');
title('Option values and greeks at 5 months to maturity');
xlabel('stock price');
ylabel('values and derivatives');
fig2 = figure;
plot(s,f(:,4),s,theta(:,4),s,delta(:,4),s,gamma(:,4),s,lambda(:,4),s,rho(:,4));
legend('value','theta','delta','gamma','lambda','rho','Location','NorthWest');
title('Option values and greeks at 3.5 months to maturity');
xlabel('stock price');
ylabel('values and derivatives');

function print_greek(ns,nt,tmat,s,t,grname,greek)

  fprintf('\n%s\n\n',grname);
  fprintf('  Stock Price  |   Time to Maturity (months)\n');
  fprintf('%16s %12.4e%12.4e%12.4e%12.4e\n', '|', 12*(tmat-t));
  fprintf('%15s+%48s\n', '---------------', ...
          '------------------------------------------------');
  for i = 1:ns
    fprintf('%12.4e%4s %12.4e%12.4e%12.4e%12.4e\n', s(i), '|', greek(i,:));
  end
```

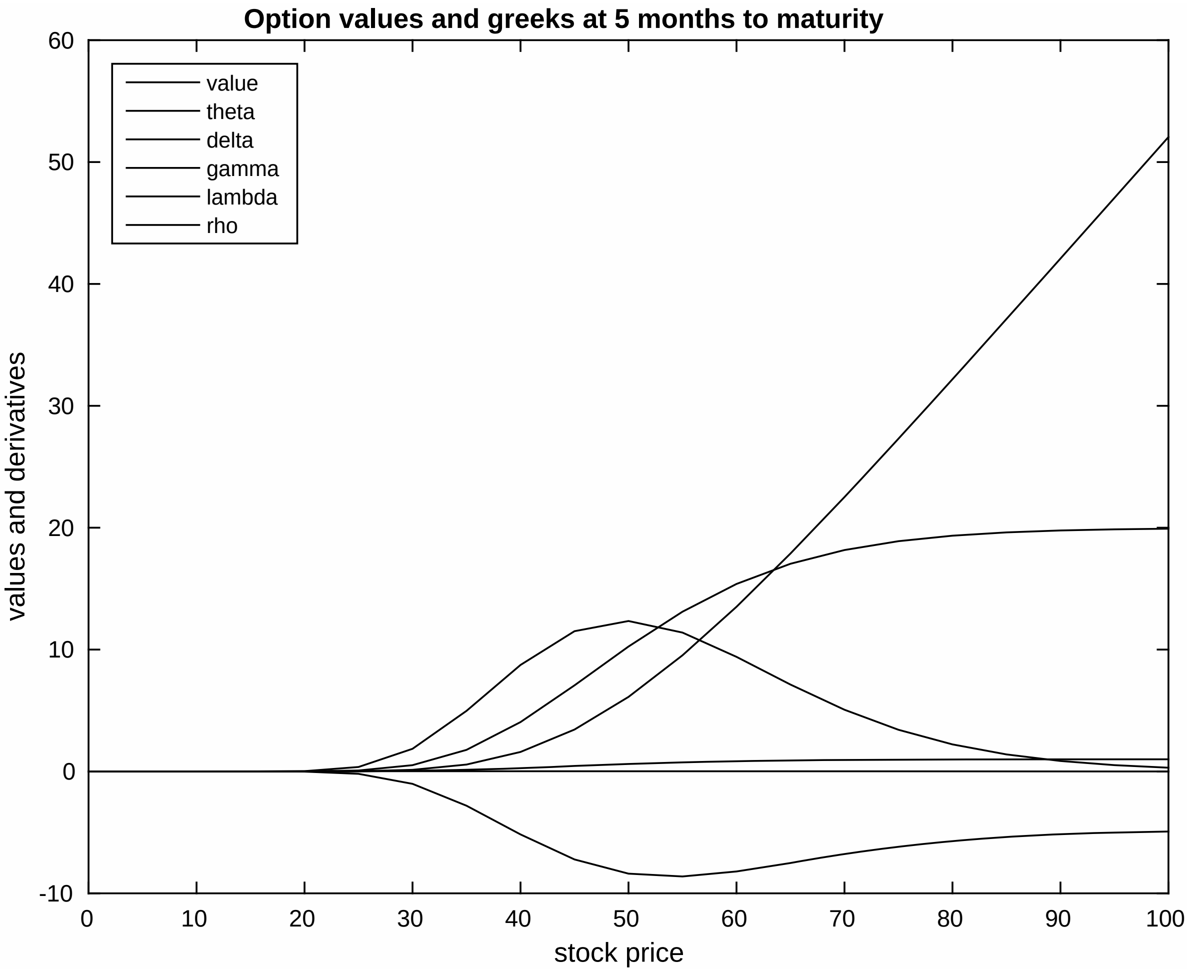
9.2 Program Results

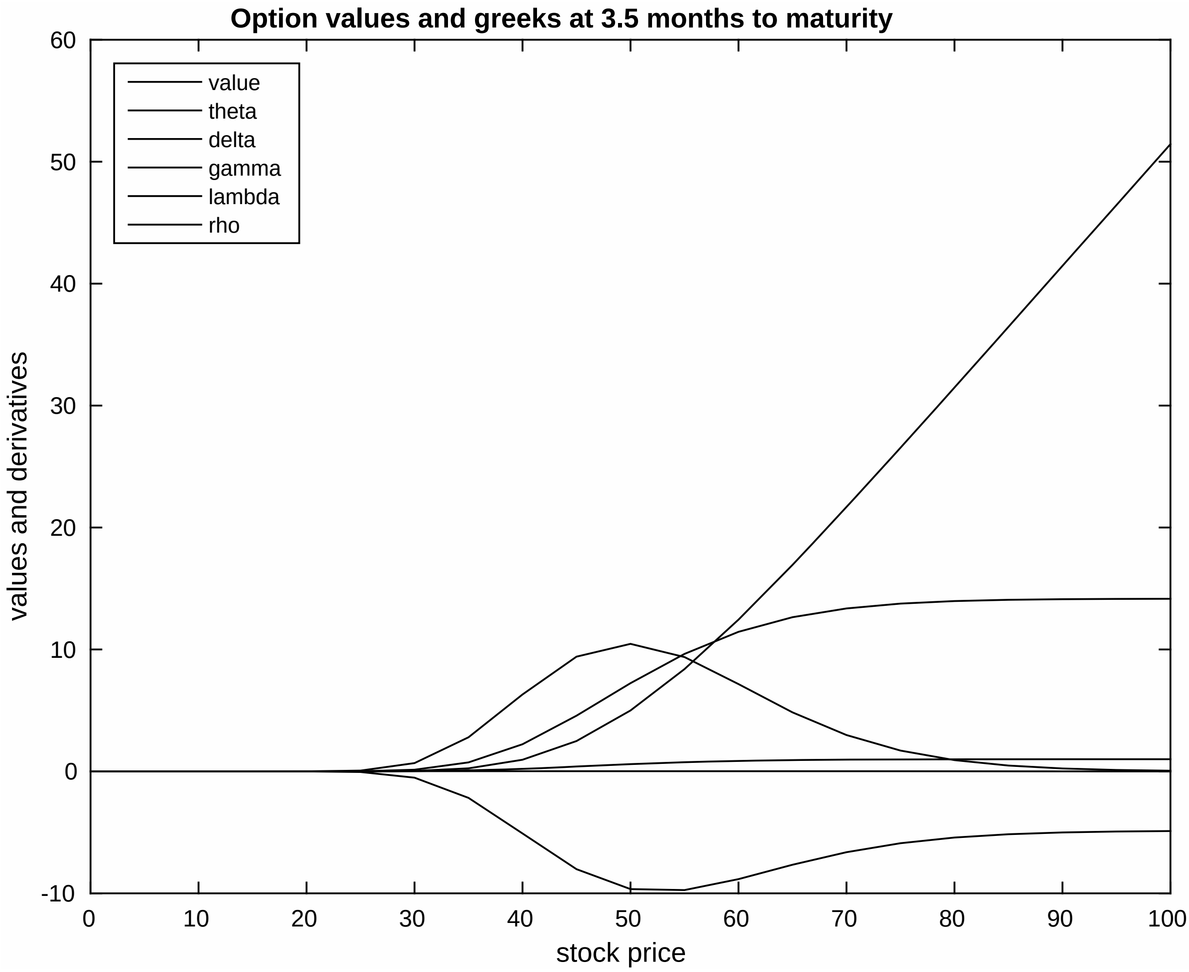
```
     d03nd example results

Option Values

  Stock Price  |   Time to Maturity (months)
               |   5.0000e+00  4.5000e+00  4.0000e+00  3.5000e+00
---------------+------------------------------------------------
  0.0000e+00   |   0.0000e+00  0.0000e+00  0.0000e+00  0.0000e+00
  5.0000e+00   |   4.4491e-19  4.5989e-21  1.5461e-23  1.0478e-26
  1.0000e+01   |   5.5566e-10  5.5129e-11  3.1298e-12  8.0281e-14
  1.5000e+01   |   4.7337e-06  1.2187e-06  2.2774e-07  2.7003e-08
  2.0000e+01   |   7.2236e-04  3.1054e-04  1.1005e-04  2.9678e-05
  2.5000e+01   |   1.6557e-02  9.6610e-03  5.0099e-03  2.2012e-03
  3.0000e+01   |   1.3307e-01  9.4037e-02  6.1869e-02  3.6848e-02
  3.5000e+01   |   5.6631e-01  4.5257e-01  3.4667e-01  2.5053e-01
  4.0000e+01   |   1.6004e+00  1.3850e+00  1.1699e+00  9.5640e-01
  4.5000e+01   |   3.4384e+00  3.1328e+00  2.8168e+00  2.4891e+00
  5.0000e+01   |   6.1165e+00  5.7600e+00  5.3874e+00  4.9960e+00
  5.5000e+01   |   9.5300e+00  9.1645e+00  8.7846e+00  8.3882e+00
  6.0000e+01   |   1.3509e+01  1.3163e+01  1.2808e+01  1.2445e+01
  6.5000e+01   |   1.7883e+01  1.7568e+01  1.7251e+01  1.6932e+01
  7.0000e+01   |   2.2513e+01  2.2230e+01  2.1949e+01  2.1671e+01
  7.5000e+01   |   2.7301e+01  2.7045e+01  2.6792e+01  2.6544e+01
```

```
8.0000e+01   |   3.2182e+01  3.1946e+01  3.1713e+01  3.1485e+01
8.5000e+01   |   3.7117e+01  3.6894e+01  3.6674e+01  3.6458e+01
9.0000e+01   |   4.2081e+01  4.1868e+01  4.1656e+01  4.1446e+01
9.5000e+01   |   4.7062e+01  4.6854e+01  4.6647e+01  4.6441e+01
1.0000e+02   |   5.2052e+01  5.1847e+01  5.1643e+01  5.1439e+01
```




Option values and greeks at 3.5 months to maturity
value
theta
delta
gamma
lambda
rho
values and derivatives
stock price
60
50
40
30
20
10
0
-10
0
10
20
30
40
50
60
70
80
90
100