NAG Library Routine Document

F04BHF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of **bold italicised** terms and other implementation-dependent details.

1   Purpose

F04BHF computes the solution to a real system of linear equations $AX = B$, where A is an n by n symmetric matrix and X and B are n by r matrices. An estimate of the condition number of A and an error bound for the computed solution are also returned.

2   Specification

```
SUBROUTINE F04BHF (UPLO, N, NRHS, A, LDA, IPIV, B, LDB, RCOND, ERRBND,    &
                   IFAIL)
INTEGER            N, NRHS, LDA, IPIV(N), LDB, IFAIL
REAL (KIND=nag_wp) A(LDA,*), B(LDB,*), RCOND, ERRBND
CHARACTER(1)       UPLO
```

3   Description

The diagonal pivoting method is used to factor A as $A = UDU^{\mathrm{T}}$, if UPLO = 'U', or $A = LDL^{\mathrm{T}}$, if UPLO = 'L', where U (or L) is a product of permutation and unit upper (lower) triangular matrices, and D is symmetric and block diagonal with 1 by 1 and 2 by 2 diagonal blocks. The factored form of A is then used to solve the system of equations $AX = B$.

4   References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia http://www.netlib.org/lapack/lug

Higham N J (2002) *Accuracy and Stability of Numerical Algorithms* (2nd Edition) SIAM, Philadelphia

5   Arguments

1:    UPLO – CHARACTER(1)   *Input*

On entry: if UPLO = 'U', the upper triangle of the matrix A is stored.

If UPLO = 'L', the lower triangle of the matrix A is stored.

Constraint: UPLO = 'U' or 'L'.

2:    N – INTEGER   *Input*

On entry: the number of linear equations n, i.e., the order of the matrix A.

Constraint: N $\geq$ 0.

3:    NRHS – INTEGER   *Input*

On entry: the number of right-hand sides r, i.e., the number of columns of the matrix B.

Constraint: NRHS $\geq$ 0.

4:      A(LDA, ∗) – REAL (KIND=nag_wp) array   *Input/Output*

Note: the second dimension of the array A must be at least max(1, N).

On entry: the n by n symmetric matrix A.

If UPLO = 'U', the leading N by N upper triangular part of the array A contains the upper triangular part of the matrix A, and the strictly lower triangular part of A is not referenced.

If UPLO = 'L', the leading N by N lower triangular part of the array A contains the lower triangular part of the matrix A, and the strictly upper triangular part of A is not referenced.

On exit: if IFAIL ≥ 0, the block diagonal matrix D and the multipliers used to obtain the factor U or L from the factorization $A = UDU^{\mathrm{T}}$ or $A = LDL^{\mathrm{T}}$ as computed by F07MDF (DSYTRF).

5:      LDA – INTEGER   *Input*

On entry: the first dimension of the array A as declared in the (sub)program from which F04BHF is called.

Constraint: LDA ≥ max(1, N).

6:      IPIV(N) – INTEGER array   *Output*

On exit: if IFAIL ≥ 0, details of the interchanges and the block structure of D, as determined by F07MDF (DSYTRF).

IPIV(k) > 0
        Rows and columns k and IPIV(k) were interchanged, and d_{kk} is a 1 by 1 diagonal block.

UPLO = 'U' and IPIV(k) = IPIV($k-1$) < 0
        Rows and columns $k-1$ and $-$IPIV(k) were interchanged and $d_{k-1:k,k-1:k}$ is a 2 by 2 diagonal block.

UPLO = 'L' and IPIV(k) = IPIV($k+1$) < 0
        Rows and columns $k+1$ and $-$IPIV(k) were interchanged and $d_{k:k+1,k:k+1}$ is a 2 by 2 diagonal block.

7:      B(LDB, ∗) – REAL (KIND=nag_wp) array   *Input/Output*

Note: the second dimension of the array B must be at least max(1, NRHS).

On entry: the n by r matrix of right-hand sides B.

On exit: if IFAIL = 0 or N + 1, the n by r solution matrix X.

8:      LDB – INTEGER   *Input*

On entry: the first dimension of the array B as declared in the (sub)program from which F04BHF is called.

Constraint: LDB ≥ max(1, N).

9:      RCOND – REAL (KIND=nag_wp)   *Output*

On exit: if no constraints are violated, an estimate of the reciprocal of the condition number of the matrix A, computed as RCOND $= 1/\left(\|A\|_1 \|A^{-1}\|_1\right)$.

10:     ERRBND – REAL (KIND=nag_wp)   *Output*

On exit: if IFAIL = 0 or N + 1, an estimate of the forward error bound for a computed solution $\hat{x}$, such that $\|\hat{x} - x\|_1 / \|x\|_1 \le$ ERRBND, where $\hat{x}$ is a column of the computed solution returned in the array B and x is the corresponding column of the exact solution X. If RCOND is less than **machine precision**, then ERRBND is returned as unity.

11:    IFAIL – INTEGER   *Input/Output*

On entry: IFAIL must be set to 0, −1 or 1. If you are unfamiliar with this argument you should refer to Section 3.4 in How to Use the NAG Library and its Documentation for details.

For environments where it might be inappropriate to halt program execution when an error is detected, the value −1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this argument, the recommended value is 0. **When the value −1 or 1 is used it is essential to test the value of IFAIL on exit.**

On exit: IFAIL = 0 unless the routine detects an error or a warning has been flagged (see Section 6).

6   Error Indicators and Warnings

If on entry IFAIL = 0 or −1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL > 0 and IFAIL ≤ N

   Diagonal block $\langle value \rangle$ of the block diagonal matrix is zero. The factorization has been completed, but the solution could not be computed.

IFAIL = N + 1

   A solution has been computed, but RCOND is less than *machine precision* so that the matrix A is numerically singular.

IFAIL = −1

   On entry, UPLO not one of `U' or `u' or `L' or `l': UPLO = $\langle value \rangle$.

IFAIL = −2

   On entry, N = $\langle value \rangle$.
   Constraint: N ≥ 0.

IFAIL = −3

   On entry, NRHS = $\langle value \rangle$.
   Constraint: NRHS ≥ 0.

IFAIL = −5

   On entry, LDA = $\langle value \rangle$ and N = $\langle value \rangle$.
   Constraint: LDA ≥ max(1, N).

IFAIL = −8

   On entry, LDB = $\langle value \rangle$ and N = $\langle value \rangle$.
   Constraint: LDB ≥ max(1, N).

IFAIL = −99

   An unexpected error has been triggered by this routine. Please contact NAG.

   See Section 3.9 in How to Use the NAG Library and its Documentation for further information.

IFAIL = −399

   Your licence key may have expired or may not have been installed correctly.

   See Section 3.8 in How to Use the NAG Library and its Documentation for further information.

IFAIL $= -999$

Dynamic memory allocation failed.

The integer allocatable memory required is N, and the real allocatable memory required is max$(2 \times N, LWORK)$, where LWORK is the optimum workspace required by F07MAF (DSYSV). If this failure occurs it may be possible to solve the equations by calling the packed storage version of F04BHF, F04BJF, or by calling F07MAF (DSYSV) directly with less than the optimum workspace (see Chapter F07).

See Section 3.7 in How to Use the NAG Library and its Documentation for further information.

7 Accuracy

The computed solution for a single right-hand side, $\hat{x}$, satisfies an equation of the form

$$(A + E)\hat{x} = b,$$

where

$$\|E\|_1 = O(\epsilon)\|A\|_1$$

and ϵ is the **machine precision**. An approximate error bound for the computed solution is given by

$$\frac{\|\hat{x} - x\|_1}{\|x\|_1} \le \kappa(A)\frac{\|E\|_1}{\|A\|_1},$$

where $\kappa(A) = \|A^{-1}\|_1\|A\|_1$, the condition number of A with respect to the solution of the linear equations. F04BHF uses the approximation $\|E\|_1 = \epsilon\|A\|_1$ to estimate ERRBND. See Section 4.4 of Anderson *et al.* (1999) for further details.

8 Parallelism and Performance

F04BHF makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the X06 Chapter Introduction for information on how to control and interrogate the OpenMP environment used within this routine. Please also consult the Users' Note for your implementation for any additional implementation-specific information.

9 Further Comments

The total number of floating-point operations required to solve the equations $AX = B$ is proportional to $\left(\frac{1}{3}n^3 + 2n^2r\right)$. The condition number estimation typically requires between four and five solves and never more than eleven solves, following the factorization.

In practice the condition number estimator is very reliable, but it can underestimate the true condition number; see Section 15.3 of Higham (2002) for further details.

The complex analogues of F04BHF are F04CHF for complex Hermitian matrices, and F04DHF for complex symmetric matrices.

10 Example

This example solves the equations

$$AX = B,$$

where A is the symmetric indefinite matrix

$$A = \begin{pmatrix} -1.81 & 2.06 & 0.63 & -1.15 \\ 2.06 & 1.15 & 1.87 & 4.20 \\ 0.63 & 1.87 & -0.21 & 3.87 \\ -1.15 & 4.20 & 3.87 & 2.07 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.96 & 3.93 \\ 6.07 & 19.25 \\ 8.38 & 9.90 \\ 9.50 & 27.85 \end{pmatrix}.$$

An estimate of the condition number of A and an approximate error bound for the computed solutions are also printed.

10.1 Program Text

```
    Program f04bhfe

!       F04BHF Example Program Text

!       Mark 26 Release. NAG Copyright 2016.

!       .. Use Statements ..
        Use nag_library, Only: f04bhf, nag_wp, x04caf
!       .. Implicit None Statement ..
        Implicit None
!       .. Parameters ..
        Integer, Parameter              :: nin = 5, nout = 6
!       .. Local Scalars ..
        Real (Kind=nag_wp)              :: errbnd, rcond
        Integer                         :: i, ierr, ifail, lda, ldb, n, nrhs
!       .. Local Arrays ..
        Real (Kind=nag_wp), Allocatable :: a(:,:), b(:,:)
        Integer, Allocatable            :: ipiv(:)
!       .. Executable Statements ..
        Write (nout,*) 'F04BHF Example Program Results'
        Write (nout,*)
        Flush (nout)
!       Skip heading in data file
        Read (nin,*)
        Read (nin,*) n, nrhs
        lda = n
        ldb = n
        Allocate (a(lda,n),b(ldb,nrhs),ipiv(n))
!       Read the upper triangular part of A from data file
        Read (nin,*)(a(i,i:n),i=1,n)

!       Read B from data file
        Read (nin,*)(b(i,1:nrhs),i=1,n)

!       Solve the equations AX = B for X

!       ifail: behaviour on error exit
!               =0 for hard exit, =1 for quiet-soft, =-1 for noisy-soft
        ifail = 1
        Call f04bhf('Upper',n,nrhs,a,lda,ipiv,b,ldb,rcond,errbnd,ifail)

        If (ifail==0) Then
!         Print solution, estimate of condition number and approximate
!         error bound

          ierr = 0
          Call x04caf('General',' ',n,nrhs,b,ldb,'Solution',ierr)

          Write (nout,*)
          Write (nout,*) 'Estimate of condition number'
          Write (nout,99999) 1.0E0_nag_wp/rcond
          Write (nout,*)
          Write (nout,*) 'Estimate of error bound for computed solutions'
          Write (nout,99999) errbnd
        Else If (ifail==n+1) Then
!         Matrix A is numerically singular.  Print estimate of
!         reciprocal of condition number and solution
          Write (nout,*)
```

```
        Write (nout,*) 'Estimate of reciprocal of condition number'
        Write (nout,99999) rcond
        Write (nout,*)
        Flush (nout)

        ierr = 0
        Call x04caf('General',' ',n,nrhs,b,ldb,'Solution',ierr)

      Else If (ifail>0 .And. ifail<=n) Then
!       The upper triangular matrix U is exactly singular.  Print
!       details of factorization
        Write (nout,*)
        Flush (nout)

        ierr = 0
        Call x04caf('Upper','Non-unit diagonal',n,n,a,lda,              &
          'Details of factorization',ierr)

!       Print pivot indices
        Write (nout,*)
        Write (nout,*) 'Pivot indices'
        Write (nout,99998) ipiv(1:n)
      Else
        Write (nout,99997) ifail
      End If

99999 Format (6X,1P,E9.1)
99998 Format ((1X,7I11))
99997 Format (1X,' ** F04BHF returned with IFAIL = ',I5)
    End Program f04bhfe
```

10.2 Program Data

```
F04BHF Example Program Data

  4     2                    : n, nrhs

 -1.81   2.06   0.63  -1.15
         1.15   1.87   4.20
               -0.21   3.87
                       2.07 : matrix A

  0.96   3.93
  6.07  19.25
  8.38   9.90
  9.50  27.85              : matrix B
```

10.3 Program Results

```
F04BHF Example Program Results

 Solution
            1          2
 1    -5.0000     2.0000
 2    -2.0000     3.0000
 3     1.0000     4.0000
 4     4.0000     1.0000

 Estimate of condition number
       7.6E+01

 Estimate of error bound for computed solutions
       8.4E-15
```