NAG Library Routine Document

D05AAF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of **bold italicised** terms and other implementation-dependent details.

1   Purpose

D05AAF solves a linear, nonsingular Fredholm equation of the second kind with a split kernel.

2   Specification

```
SUBROUTINE D05AAF (LAMBDA, A, B, K1, K2, G, F, C, N, IND, W1, W2, WD,    &
                   LDW1, LDW2, IFAIL)

INTEGER           N, IND, LDW1, LDW2, IFAIL
REAL (KIND=nag_wp) LAMBDA, A, B, K1, K2, G, F(N), C(N), W1(LDW1,LDW2),   &
                   W2(LDW2,4), WD(LDW2)
EXTERNAL          K1, K2, G
```

3   Description

D05AAF solves an integral equation of the form

$$f(x) - \lambda \int_a^b k(x,s) f(s)\, ds = g(x)$$

for $a \le x \le b$, when the kernel k is defined in two parts: $k = k_1$ for $a \le s \le x$ and $k = k_2$ for $x < s \le b$. The method used is that of El–Gendi (1969) for which, it is important to note, each of the functions k_1 and k_2 must be defined, smooth and nonsingular, for all x and s in the interval $[a,b]$.

An approximation to the solution $f(x)$ is found in the form of an n term Chebyshev series $\sum_{i=1}^{n} {}' c_i T_i(x)$, where $'$ indicates that the first term is halved in the sum. The coefficients c_i, for $i = 1, 2, \ldots, n$, of this series are determined directly from approximate values f_i, for $i = 1, 2, \ldots, n$, of the function $f(x)$ at the first n of a set of $m + 1$ Chebyshev points:

$$x_i = \tfrac{1}{2}(a + b + (b - a)\cos[(i-1)\pi/m]), \quad i = 1, 2, \ldots, m + 1.$$

The values f_i are obtained by solving simultaneous linear algebraic equations formed by applying a quadrature formula (equivalent to the scheme of Clenshaw and Curtis (1960)) to the integral equation at the above points.

In general $m = n - 1$. However, if the kernel k is centro-symmetric in the interval $[a,b]$, i.e., if $k(x,s) = k(a + b - x, a + b - s)$, then the routine is designed to take advantage of this fact in the formation and solution of the algebraic equations. In this case, symmetry in the function $g(x)$ implies symmetry in the function $f(x)$. In particular, if $g(x)$ is even about the mid-point of the range of integration, then so also is $f(x)$, which may be approximated by an even Chebyshev series with $m = 2n - 1$. Similarly, if $g(x)$ is odd about the mid-point then $f(x)$ may be approximated by an odd series with $m = 2n$.

4   References

Clenshaw C W and Curtis A R (1960) A method for numerical integration on an automatic computer *Numer. Math.* **2** 197–205

El–Gendi S E (1969) Chebyshev solution of differential, integral and integro-differential equations *Comput. J.* **12** 282–287

5 Arguments

1:    LAMBDA – REAL (KIND=nag_wp)   *Input*

   On entry: the value of the parameter λ of the integral equation.

2:    A – REAL (KIND=nag_wp)   *Input*

   On entry: a, the lower limit of integration.

3:    B – REAL (KIND=nag_wp)   *Input*

   On entry: b, the upper limit of integration.

   Constraint: B > A.

4:    K1 – REAL (KIND=nag_wp) FUNCTION, supplied by the user.   *External Procedure*

   K1 must evaluate the kernel $k(x,s) = k_1(x,s)$ of the integral equation for $a \le s \le x$.

   ```
   The specification of K1 is:

   FUNCTION K1 (X, S)
   REAL (KIND=nag_wp) K1

   REAL (KIND=nag_wp) X, S
   ```

   1:    X – REAL (KIND=nag_wp)   *Input*
   2:    S – REAL (KIND=nag_wp)   *Input*

   On entry: the values of x and s at which $k_1(x,s)$ is to be evaluated.

   K1 must either be a module subprogram USEd by, or declared as EXTERNAL in, the (sub) program from which D05AAF is called. Arguments denoted as *Input* must **not** be changed by this procedure.

5:    K2 – REAL (KIND=nag_wp) FUNCTION, supplied by the user.   *External Procedure*

   K2 must evaluate the kernel $k(x,s) = k_2(x,s)$ of the integral equation for $x < s \le b$.

   ```
   The specification of K2 is:

   FUNCTION K2 (X, S)
   REAL (KIND=nag_wp) K2

   REAL (KIND=nag_wp) X, S
   ```

   1:    X – REAL (KIND=nag_wp)   *Input*
   2:    S – REAL (KIND=nag_wp)   *Input*

   On entry: the values of x and s at which $k_2(x,s)$ is to be evaluated.

   K2 must either be a module subprogram USEd by, or declared as EXTERNAL in, the (sub) program from which D05AAF is called. Arguments denoted as *Input* must **not** be changed by this procedure.

   Note that the functions k_1 and k_2 must be defined, smooth and nonsingular for all x and s in the interval $[a, b]$.

6:    G – REAL (KIND=nag_wp) FUNCTION, supplied by the user.   *External Procedure*

   G must evaluate the function $g(x)$ for $a \le x \le b$.

   ```
   The specification of G is:

   FUNCTION G (X)
   ```

```
REAL (KIND=nag_wp) G
REAL (KIND=nag_wp) X
```

1:     X – REAL (KIND=nag_wp)   *Input*

   On entry: the values of x at which $g(x)$ is to be evaluated.

G must either be a module subprogram USEd by, or declared as EXTERNAL in, the (sub) program from which D05AAF is called. Arguments denoted as *Input* must **not** be changed by this procedure.

7:     F(N) – REAL (KIND=nag_wp) array   *Output*

   On exit: the approximate values f_i, for $i = 1, 2, \ldots, N$, of $f(x)$ evaluated at the first N of $m + 1$ Chebyshev points x_i, (see Section 3).

   If IND = 0 or 3, $m = N - 1$.

   If IND = 1, $m = 2 \times N$.

   If IND = 2, $m = 2 \times N - 1$.

8:     C(N) – REAL (KIND=nag_wp) array   *Output*

   On exit: the coefficients c_i, for $i = 1, 2, \ldots, N$, of the Chebyshev series approximation to $f(x)$.

   If IND = 1 this series contains polynomials of odd order only and if IND = 2 the series contains even order polynomials only.

9:     N – INTEGER   *Input*

   On entry: the number of terms in the Chebyshev series required to approximate $f(x)$.

   Constraint: N $\geq$ 1.

10:    IND – INTEGER   *Input*

   On entry: determines the forms of the kernel, $k(x, s)$, and the function $g(x)$.

   IND = 0
      $k(x, s)$ is not centro-symmetric (or no account is to be taken of centro-symmetry).

   IND = 1
      $k(x, s)$ is centro-symmetric and $g(x)$ is odd.

   IND = 2
      $k(x, s)$ is centro-symmetric and $g(x)$ is even.

   IND = 3
      $k(x, s)$ is centro-symmetric but $g(x)$ is neither odd nor even.

   Constraint: IND = 0, 1, 2 or 3.

11:    W1(LDW1, LDW2) – REAL (KIND=nag_wp) array   *Workspace*
12:    W2(LDW2, 4) – REAL (KIND=nag_wp) array   *Workspace*
13:    WD(LDW2) – REAL (KIND=nag_wp) array   *Workspace*

14:    LDW1 – INTEGER   *Input*

   On entry: the first dimension of the array W1 as declared in the (sub)program from which D05AAF is called.

   Constraint: LDW1 $\geq$ N.

15:     LDW2 – INTEGER   *Input*

On entry: the second dimension of the array W1 and the first dimension of the array W2 and the dimension of the array WD as declared in the (sub)program from which D05AAF is called.

Constraint: $\text{LDW2} \geq 2 \times \text{N} + 2$.

16:     IFAIL – INTEGER   *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this argument you should refer to Section 3.4 in How to Use the NAG Library and its Documentation for details.

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this argument, the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

On exit: $\text{IFAIL} = 0$ unless the routine detects an error or a warning has been flagged (see Section 6).

6   Error Indicators and Warnings

If on entry $\text{IFAIL} = 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

$\text{IFAIL} = 1$

On entry, $\text{A} \geq \text{B}$ or $\text{N} < 1$.

$\text{IFAIL} = 2$

A failure has occurred due to proximity to an eigenvalue. In general, if LAMBDA is near an eigenvalue of the integral equation, the corresponding matrix will be nearly singular. In the special case, $m = 1$, the matrix reduces to a zero-valued number.

$\text{IFAIL} = -99$

An unexpected error has been triggered by this routine. Please contact NAG.

See Section 3.9 in How to Use the NAG Library and its Documentation for further information.

$\text{IFAIL} = -399$

Your licence key may have expired or may not have been installed correctly.

See Section 3.8 in How to Use the NAG Library and its Documentation for further information.

$\text{IFAIL} = -999$

Dynamic memory allocation failed.

See Section 3.7 in How to Use the NAG Library and its Documentation for further information.

7   Accuracy

No explicit error estimate is provided by the routine but it is usually possible to obtain a good indication of the accuracy of the solution either

(i)   by examining the size of the later Chebyshev coefficients c_i, or

(ii) by comparing the coefficients c_i or the function values f_i for two or more values of N.

8 Parallelism and Performance

D05AAF is threaded by NAG for parallel execution in multithreaded implementations of the NAG Library.

D05AAF makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the X06 Chapter Introduction for information on how to control and interrogate the OpenMP environment used within this routine. Please also consult the Users' Note for your implementation for any additional implementation-specific information.

9 Further Comments

The time taken by D05AAF increases with N.

This routine may be used to solve an equation with a continuous kernel by defining K1 and K2 to be identical.

This routine may also be used to solve a Volterra equation by defining K2 (or K1) to be identically zero.

10 Example

This example solves the equation

$$f(x) - \int_0^1 k(x, s) f(s) \, ds = \left(1 - \frac{1}{\pi^2} \right) \sin(\pi x)$$

where

$$k(x, s) = \begin{cases} s(1 - x) & \text{for } 0 \le s \le x, \\ x(1 - s) & \text{for } x < s \le 1. \end{cases}$$

Five terms of the Chebyshev series are sought, taking advantage of the centro-symmetry of the $k(x, s)$ and even nature of $g(x)$ about the mid-point of the range $[0, 1]$.

The approximate solution at the point $x = 0.1$ is calculated by calling C06DCF.

10.1 Program Text

```
!   D05AAF Example Program Text
!   Mark 26 Release. NAG Copyright 2016.

    Module d05aafe_mod

!     D05AAF Example Program Module:
!            Parameters and User-defined Routines

!     .. Use Statements ..
      Use nag_library, Only: nag_wp
!     .. Implicit None Statement ..
      Implicit None
!     .. Accessibility Statements ..
      Private
      Public                                 :: g, k1, k2
!     .. Parameters ..
      Real (Kind=nag_wp), Parameter, Public :: a = 0.0_nag_wp
      Real (Kind=nag_wp), Parameter, Public :: b = 1.0_nag_wp
      Real (Kind=nag_wp), Parameter, Public :: lambda = 1.0_nag_wp
      Real (Kind=nag_wp), Parameter, Public :: xval = 0.1_nag_wp
      Integer, Parameter, Public            :: ind = 2, n = 5, nout = 6
      Integer, Parameter, Public            :: ldw1 = n
      Integer, Parameter, Public            :: ldw2 = 2*n + 2
    Contains
```

```
      Function k1(x,s)

!       .. Function Return Value ..
        Real (Kind=nag_wp)              :: k1
!       .. Scalar Arguments ..
        Real (Kind=nag_wp), Intent (In) :: s, x
!       .. Executable Statements ..
        k1 = s*(1.0_nag_wp-x)

        Return
      End Function k1
      Function k2(x,s)

!       .. Function Return Value ..
        Real (Kind=nag_wp)              :: k2
!       .. Scalar Arguments ..
        Real (Kind=nag_wp), Intent (In) :: s, x
!       .. Executable Statements ..
        k2 = x*(1.0_nag_wp-s)

        Return
      End Function k2
      Function g(x)

!       .. Use Statements ..
        Use nag_library, Only: x01aaf
!       .. Function Return Value ..
        Real (Kind=nag_wp)             :: g
!       .. Scalar Arguments ..
        Real (Kind=nag_wp), Intent (In) :: x
!       .. Local Scalars ..
        Real (Kind=nag_wp)             :: pi
!       .. Intrinsic Procedures ..
        Intrinsic                      :: sin
!       .. Executable Statements ..
        pi = x01aaf(pi)
        g = sin(pi*x)*(1.0_nag_wp-1.0_nag_wp/(pi*pi))

        Return
      End Function g
    End Module d05aafe_mod
    Program d05aafe

!     D05AAF Example Main Program

!     .. Use Statements ..
      Use nag_library, Only: c06dcf, d05aaf, nag_wp
      Use d05aafe_mod, Only: a, b, g, ind, k1, k2, lambda, ldw1, ldw2, n,      &
                             nout, xval
!     .. Implicit None Statement ..
      Implicit None
!     .. Local Scalars ..
      Integer                          :: ifail, is
!     .. Local Arrays ..
      Real (Kind=nag_wp)               :: ans(1), x(1)
      Real (Kind=nag_wp), Allocatable  :: c(:), f(:), w1(:,:), w2(:,:), wd(:)
!     .. Executable Statements ..
      Write (nout,*) 'D05AAF Example Program Results'

      Allocate (c(n),f(n),w1(ldw1,ldw2),w2(ldw2,4),wd(ldw2))

      ifail = 0
      Call d05aaf(lambda,a,b,k1,k2,g,f,c,n,ind,w1,w2,wd,ldw1,ldw2,ifail)

      Write (nout,99999)
      Write (nout,99998) c(1:n)

      x(1) = xval

      Select Case (ind)
      Case (1)
```

```
      is = 3
    Case (2)
      is = 2
    Case Default
      is = 1
    End Select

    ifail = 0
    Call c06dcf(x,1,a,b,c,n,is,ans,ifail)

    Write (nout,99997) 'X=', x, '    ANS=', ans

99999 Format (/,1X,'Kernel is centro-symmetric and G is even so the ',          &
      'solution is even',/,/,/,1X,'Chebyshev coefficients',/)
99998 Format (1X,5E14.4,/)
99997 Format (1X,A,F5.2,A,1F10.4)
    End Program d05aafe
```

10.2 Program Data

None.

10.3 Program Results

```
 D05AAF Example Program Results

 Kernel is centro-symmetric and G is even so the solution is even

 Chebyshev coefficients

     0.9440E+00   -0.4994E+00    0.2799E-01   -0.5967E-03    0.6658E-05

 X= 0.10    ANS=    0.3090
```