

## NAG Library Routine Document

### F08QUF (ZTRSEN)

**Note:** before using this routine, please read the Users' Note for your implementation to check the interpretation of *bold italicised* terms and other implementation-dependent details.

#### 1 Purpose

F08QUF (ZTRSEN) reorders the Schur factorization of a complex general matrix so that a selected cluster of eigenvalues appears in the leading elements on the diagonal of the Schur form. The routine also optionally computes the reciprocal condition numbers of the cluster of eigenvalues and/or the invariant subspace.

#### 2 Specification

SUBROUTINE F08QUF (JOB, COMPQ, SELECT, N, T, LDT, Q, LDQ, W, M, S, SEP, &  
WORK, LWORK, INFO)

INTEGER N, LDT, LDQ, M, LWORK, INFO  
REAL (KIND=nag\_wp) S, SEP  
COMPLEX (KIND=nag\_wp) T(LDT,\*), Q(LDQ,\*), W(\*), WORK(max(1,LWORK))  
LOGICAL SELECT(\*)  
CHARACTER(1) JOB, COMPQ

The routine may be called by its LAPACK name *ztrsen*.

#### 3 Description

F08QUF (ZTRSEN) reorders the Schur factorization of a complex general matrix  $A = QTQ^H$ , so that a selected cluster of eigenvalues appears in the leading diagonal elements of the Schur form.

The reordered Schur form  $\tilde{T}$  is computed by a unitary similarity transformation:  $\tilde{T} = Z^H T Z$ . Optionally the updated matrix  $\tilde{Q}$  of Schur vectors is computed as  $\tilde{Q} = QZ$ , giving  $A = \tilde{Q}\tilde{T}\tilde{Q}^H$ .

Let  $\tilde{T} = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ , where the selected eigenvalues are precisely the eigenvalues of the leading  $m$  by  $m$  sub-matrix  $T_{11}$ . Let  $\tilde{Q}$  be correspondingly partitioned as  $(Q_1 \ Q_2)$  where  $Q_1$  consists of the first  $m$  columns of  $Q$ . Then  $AQ_1 = Q_1T_{11}$ , and so the  $m$  columns of  $Q_1$  form an orthonormal basis for the invariant subspace corresponding to the selected cluster of eigenvalues.

Optionally the routine also computes estimates of the reciprocal condition numbers of the average of the cluster of eigenvalues and of the invariant subspace.

#### 4 References

Golub G H and Van Loan C F (1996) *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

#### 5 Arguments

- 1: JOB – CHARACTER(1) *Input*  
*On entry:* indicates whether condition numbers are required for the cluster of eigenvalues and/or the invariant subspace.  
 JOB = 'N'  
 No condition numbers are required.

JOB = 'E'

Only the condition number for the cluster of eigenvalues is computed.

JOB = 'V'

Only the condition number for the invariant subspace is computed.

JOB = 'B'

Condition numbers for both the cluster of eigenvalues and the invariant subspace are computed.

*Constraint:* JOB = 'N', 'E', 'V' or 'B'.

2: COMPQ – CHARACTER(1) *Input*

*On entry:* indicates whether the matrix  $Q$  of Schur vectors is to be updated.

COMPQ = 'V'

The matrix  $Q$  of Schur vectors is updated.

COMPQ = 'N'

No Schur vectors are updated.

*Constraint:* COMPQ = 'V' or 'N'.

3: SELECT(\*) – LOGICAL array *Input*

**Note:** the dimension of the array SELECT must be at least  $\max(1, N)$ .

*On entry:* specifies the eigenvalues in the selected cluster. To select a complex eigenvalue  $\lambda_j$ , SELECT( $j$ ) must be set .TRUE..

4: N – INTEGER *Input*

*On entry:*  $n$ , the order of the matrix  $T$ .

*Constraint:*  $N \geq 0$ .

5: T(LDT, \*) – COMPLEX (KIND=nag\_wp) array *Input/Output*

**Note:** the second dimension of the array T must be at least  $\max(1, N)$ .

*On entry:* the  $n$  by  $n$  upper triangular matrix  $T$ , as returned by F08PSF (ZHSEQR).

*On exit:* T is overwritten by the updated matrix  $\tilde{T}$ .

6: LDT – INTEGER *Input*

*On entry:* the first dimension of the array T as declared in the (sub)program from which F08QUF (ZTRSEN) is called.

*Constraint:*  $LDT \geq \max(1, N)$ .

7: Q(LDQ, \*) – COMPLEX (KIND=nag\_wp) array *Input/Output*

**Note:** the second dimension of the array Q must be at least  $\max(1, N)$  if COMPQ = 'V' and at least 1 if COMPQ = 'N'.

*On entry:* if COMPQ = 'V', Q must contain the  $n$  by  $n$  unitary matrix  $Q$  of Schur vectors, as returned by F08PSF (ZHSEQR).

*On exit:* if COMPQ = 'V', Q contains the updated matrix of Schur vectors; the first  $m$  columns of Q form an orthonormal basis for the specified invariant subspace.

If COMPQ = 'N', Q is not referenced.

- 8: LDQ – INTEGER *Input*  
*On entry:* the first dimension of the array Q as declared in the (sub)program from which F08QUF (ZTRSEN) is called.  
*Constraints:*  
 if COMPQ = 'V',  $LDQ \geq \max(1, N)$ ;  
 if COMPQ = 'N',  $LDQ \geq 1$ .
- 9: W(\*) – COMPLEX (KIND=nag\_wp) array *Output*  
**Note:** the dimension of the array W must be at least  $\max(1, N)$ .  
*On exit:* the reordered eigenvalues of  $\tilde{T}$ . The eigenvalues are stored in the same order as on the diagonal of  $\tilde{T}$ .
- 10: M – INTEGER *Output*  
*On exit:*  $m$ , the dimension of the specified invariant subspace, which is the same as the number of selected eigenvalues (see SELECT);  $0 \leq m \leq n$ .
- 11: S – REAL (KIND=nag\_wp) *Output*  
*On exit:* if JOB = 'E' or 'B', S is a lower bound on the reciprocal condition number of the average of the selected cluster of eigenvalues. If  $M = 0$  or  $N$ ,  $S = 1$ .  
 If JOB = 'N' or 'V', S is not referenced.
- 12: SEP – REAL (KIND=nag\_wp) *Output*  
*On exit:* if JOB = 'V' or 'B', SEP is the estimated reciprocal condition number of the specified invariant subspace. If  $M = 0$  or  $N$ ,  $SEP = \|T\|$ .  
 If JOB = 'N' or 'E', SEP is not referenced.
- 13: WORK(max(1, LWORK)) – COMPLEX (KIND=nag\_wp) array *Workspace*  
*On exit:* if INFO = 0, the real part of WORK(1) contains the minimum value of LWORK required for optimal performance.
- 14: LWORK – INTEGER *Input*  
*On entry:* the dimension of the array WORK as declared in the (sub)program from which F08QUF (ZTRSEN) is called, unless LWORK = -1, in which case a workspace query is assumed and the routine only calculates the minimum dimension of WORK.  
*Constraints:*  
 if JOB = 'N',  $LWORK \geq 1$  or LWORK = -1;  
 if JOB = 'E',  $LWORK \geq \max(1, m \times (N - m))$  or LWORK = -1;  
 if JOB = 'V' or 'B',  $LWORK \geq \max(1, 2m \times (N - m))$  or LWORK = -1.  
 The actual amount of workspace required cannot exceed  $N^2/4$  if JOB = 'E' or  $N^2/2$  if JOB = 'V' or 'B'.
- 15: INFO – INTEGER *Output*  
*On exit:* INFO = 0 unless the routine detects an error (see Section 6).

## 6 Error Indicators and Warnings

INFO < 0

If INFO =  $-i$ , argument  $i$  had an illegal value. An explanatory message is output, and execution of the program is terminated.

## 7 Accuracy

The computed matrix  $\tilde{T}$  is similar to a matrix  $(T + E)$ , where

$$\|E\|_2 = O(\epsilon)\|T\|_2,$$

and  $\epsilon$  is the *machine precision*.

S cannot underestimate the true reciprocal condition number by more than a factor of  $\sqrt{\min(m, n - m)}$ . SEP may differ from the true value by  $\sqrt{m(n - m)}$ . The angle between the computed invariant subspace and the true subspace is  $\frac{O(\epsilon)\|A\|_2}{sep}$ .

The values of the eigenvalues are never changed by the reordering.

## 8 Parallelism and Performance

F08QUF (ZTRSEN) makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the X06 Chapter Introduction for information on how to control and interrogate the OpenMP environment used within this routine. Please also consult the Users' Note for your implementation for any additional implementation-specific information.

## 9 Further Comments

The real analogue of this routine is F08QGF (DTRSEN).

## 10 Example

This example reorders the Schur factorization of the matrix  $A = QTQ^H$  such that the eigenvalues stored in elements  $t_{11}$  and  $t_{44}$  appear as the leading elements on the diagonal of the reordered matrix  $\tilde{T}$ , where

$$T = \begin{pmatrix} -6.0004 - 6.9999i & 0.3637 - 0.3656i & -0.1880 + 0.4787i & 0.8785 - 0.2539i \\ 0.0000 + 0.0000i & -5.0000 + 2.0060i & -0.0307 - 0.7217i & -0.2290 + 0.1313i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 7.9982 - 0.9964i & 0.9357 + 0.5359i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 3.0023 - 3.9998i \end{pmatrix}$$

and

$$Q = \begin{pmatrix} -0.8347 - 0.1364i & -0.0628 + 0.3806i & 0.2765 - 0.0846i & 0.0633 - 0.2199i \\ 0.0664 - 0.2968i & 0.2365 + 0.5240i & -0.5877 - 0.4208i & 0.0835 + 0.2183i \\ -0.0362 - 0.3215i & 0.3143 - 0.5473i & 0.0576 - 0.5736i & 0.0057 - 0.4058i \\ 0.0086 + 0.2958i & -0.3416 - 0.0757i & -0.1900 - 0.1600i & 0.8327 - 0.1868i \end{pmatrix}.$$

The original matrix  $A$  is given in Section 10 in F08NTF (ZUNGHR).

## 10.1 Program Text

```

Program f08qufe

!      F08QUF Example Program Text

!      Mark 26 Release. NAG Copyright 2016.

!      .. Use Statements ..
Use nag_library, Only: nag_wp, x02ajf, x04dbf, zgemm, zlange => f06uaf, &
                        ztrsen

!      .. Implicit None Statement ..
Implicit None

!      .. Parameters ..
Integer, Parameter      :: nin = 5, nout = 6

!      .. Local Scalars ..
Complex (Kind=nag_wp)  :: alpha, beta
Real (Kind=nag_wp)     :: norm, s, sep
Integer                :: i, ifail, info, lda, ldc, ldq, ldt, &
                        lwork, m, n

!      .. Local Arrays ..
Complex (Kind=nag_wp), Allocatable :: a(:,,:), c(:,,:), q(:,,:), t(:,,:), &
                        w(:,), work(:)
Real (Kind=nag_wp)             :: rwork(1)
Logical, Allocatable          :: select(:)
Character (1)                 :: clabs(1), rlabs(1)

!      .. Intrinsic Procedures ..
Intrinsic                    :: cmplx

!      .. Executable Statements ..
Write (nout,*) 'F08QUF Example Program Results'
Write (nout,*)
Flush (nout)

!      Skip heading in data file
Read (nin,*)
Read (nin,*) n
ldc = n
lda = n
ldq = n
ldt = n
lwork = (n*n)/2
Allocate (a(lda,n),c(ldc,n),q(ldq,n),t(ldt,n),w(n),work(lwork), &
         select(n))

!      Read T, Q and the logical array SELECT from data file

Read (nin,*)(t(i,1:n),i=1,n)
Read (nin,*)
Read (nin,*)(q(i,1:n),i=1,n)
Read (nin,*)
Read (nin,*) select(1:n)

!      Compute Q * T * Q**T to find A
!      The NAG name equivalent of zgemm is f06zaf
alpha = cmplx(1,kind=nag_wp)
beta = cmplx(0,kind=nag_wp)
Call zgemm('N','N',n,n,n,alpha,q,ldq,t,ldt,beta,c,ldc)
Call zgemm('N','C',n,n,n,alpha,c,ldc,q,ldq,beta,a,lda)

!      Print Matrix A, as computed from Q * T * Q**T
!      ifail: behaviour on error exit
!      =0 for hard exit, =1 for quiet-soft, =-1 for noisy-soft
ifail = 0
Call x04dbf('General',' ',n,n,a,lda,'Bracketed','F7.4', &
         'Matrix A created from Q*T*Q^T','Integer',rlabs,'Integer',clabs,80,0, &
         ifail)

Write (nout,*)
Flush (nout)

!      Reorder the Schur factor T and update the matrix Q to obtain TT and QT

```

```

!       The NAG name equivalent of ztrsen is f08quf
!       Call ztrsen('Both','Vectors',select,n,t,ldt,q,ldq,w,m,s,sep,work,lwork, &
!           info)

!       Compute (Q * T * Q^H) - (QT * TT * QT^H) and store in A,
!       i.e. the difference between reconstructed A using Schur and reordered
!       Schur decompositions.
!       alpha = cmplx(1,kind=nag_wp)
!       beta  = cmplx(0,kind=nag_wp)
!       Call zgemm('N','N',n,n,n,alpha,q,ldq,t,ldt,beta,c,ldc)
!       alpha = cmplx(-1,kind=nag_wp)
!       beta  = cmplx(1,kind=nag_wp)
!       Call zgemm('N','C',n,n,n,alpha,c,ldc,q,ldq,beta,a,lda)

!       Find norm of difference matrix and print warning if it is too large
!       f06uaf is the NAG name equivalent of the LAPACK auxiliary zlange
!       norm = zlange('O',lda,n,a,lda,rwork)
!       If (norm>x02ajf())**0.5_nag_wp) Then
!           Write (nout,*) 'Norm of A - (QT * TT * QT^H) is much greater than 0.'
!           Write (nout,*) 'Schur factorization has failed.'
!       Else
!           Print condition estimates
!           Write (nout,99999) 'Condition number estimate', &
!               ' of the selected cluster of eigenvalues = ', 1.0_nag_wp/s
!           Write (nout,*)
!           Write (nout,99999) 'Condition number estimate of the specified ', &
!               ' invariant subspace = ', 1.0_nag_wp/sep
!       End If

99999 Format (1X,A,A,1P,E10.2)
End Program f08qufe

```

## 10.2 Program Data

F08QUF Example Program Data

```

4
(-6.0004,-6.9999) ( 0.3637,-0.3656) (-0.1880, 0.4787) ( 0.8785,-0.2539)
( 0.0000, 0.0000) (-5.0000, 2.0060) (-0.0307,-0.7217) (-0.2290, 0.1313)
( 0.0000, 0.0000) ( 0.0000, 0.0000) ( 7.9982,-0.9964) ( 0.9357, 0.5359)
( 0.0000, 0.0000) ( 0.0000, 0.0000) ( 0.0000, 0.0000) ( 3.0023,-3.9998)
:End of matrix T
(-0.8347,-0.1364) (-0.0628, 0.3806) ( 0.2765,-0.0846) ( 0.0633,-0.2199)
( 0.0664,-0.2968) ( 0.2365, 0.5240) (-0.5877,-0.4208) ( 0.0835, 0.2183)
(-0.0362,-0.3215) ( 0.3143,-0.5473) ( 0.0576,-0.5736) ( 0.0057,-0.4058)
( 0.0086, 0.2958) (-0.3416,-0.0757) (-0.1900,-0.1600) ( 0.8327,-0.1868)
:End of matrix Q
T   F   F   T
:End of SELECT

```

## 10.3 Program Results

F08QUF Example Program Results

Matrix A created from Q\*T\*Q^T

```

1           2           3           4
1 (-3.9702,-5.0406) (-4.1108, 3.7002) (-0.3403, 1.0098) ( 1.2899,-0.8590)
2 ( 0.3397,-1.5006) ( 1.5201,-0.4301) ( 1.8797,-5.3804) ( 3.3606, 0.6498)
3 ( 3.3101,-3.8506) ( 2.4996, 3.4504) ( 0.8802,-1.0802) ( 0.6401,-1.4800)
4 (-1.0999, 0.8199) ( 1.8103,-1.5905) ( 3.2502, 1.3297) ( 1.5701,-3.4397)

```

Condition number estimate of the selected cluster of eigenvalues = 1.02E+00

Condition number estimate of the specified invariant subspace = 1.82E-01