NAG Library Routine Document

S19ACF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

S19ACF returns a value for the Kelvin function $\operatorname{ker} x$, via the function name.

2 Specification

```
FUNCTION S19ACF (X, IFAIL)
REAL (KIND=nag_wp) S19ACF
INTEGER            IFAIL
REAL (KIND=nag_wp) X
```

3 Description

S19ACF evaluates an approximation to the Kelvin function $\operatorname{ker} x$.

Note: for $x < 0$ the function is undefined and at $x = 0$ it is infinite so we need only consider $x > 0$.

The routine is based on several Chebyshev expansions:

For $0 < x \leq 1$,

$$\operatorname{ker} x = -f(t) \log x + \frac{\pi}{16} x^2 g(t) + y(t)$$

where $f(t)$, $g(t)$ and $y(t)$ are expansions in the variable $t = 2x^4 - 1$.

For $1 < x \leq 3$,

$$\operatorname{ker} x = \exp\left(-\frac{11}{16} x\right) q(t)$$

where $q(t)$ is an expansion in the variable $t = x - 2$.

For $x > 3$,

$$\operatorname{ker} x = \sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}} \left[\left(1 + \frac{1}{x} c(t)\right) \cos \beta - \frac{1}{x} d(t) \sin \beta\right]$$

where $\beta = \frac{x}{\sqrt{2}} + \frac{\pi}{8}$, and $c(t)$ and $d(t)$ are expansions in the variable $t = \frac{6}{x} - 1$.

When x is sufficiently close to zero, the result is computed as

$$\operatorname{ker} x = -\gamma - \log\left(\frac{x}{2}\right) + \left(\pi - \frac{3}{8} x^2\right) \frac{x^2}{16}$$

and when x is even closer to zero, simply as $\operatorname{ker} x = -\gamma - \log\left(\frac{x}{2}\right)$.

For large x, $\operatorname{ker} x$ is asymptotically given by $\sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}}$ and this becomes so small that it cannot be computed without underflow and the routine fails.

4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

5 Parameters

1: X – REAL (KIND=nag_wp) *Input*

On entry: the argument x of the function.

Constraint: $X > 0.0$.

2: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this parameter you should refer to Section 3.3 in the Essential Introduction for details.

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this parameter, the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

On exit: $\text{IFAIL} = 0$ unless the routine detects an error or a warning has been flagged (see Section 6).

6 Error Indicators and Warnings

If on entry $\text{IFAIL} = 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

$\text{IFAIL} = 1$

On entry, X is too large: the result underflows. On soft failure, the routine returns zero. See also the Users' Note for your implementation.

$\text{IFAIL} = 2$

On entry, $X \leq 0.0$: the function is undefined. On soft failure the routine returns zero.

7 Accuracy

Let E be the absolute error in the result, ϵ be the relative error in the result and δ be the relative error in the argument. If δ is somewhat larger than the ***machine precision***, then we have:

$$E \simeq \left| \frac{x}{\sqrt{2}} (\operatorname{ker}_1 x + \operatorname{kei}_1 x) \right| \delta,$$

$$\epsilon \simeq \left| \frac{x}{\sqrt{2}} \frac{\operatorname{ker}_1 x + \operatorname{kei}_1 x}{\operatorname{ker} x} \right| \delta.$$

For very small x, the relative error amplification factor is approximately given by $\frac{1}{|\log x|}$, which implies a strong attenuation of relative error. However, ϵ in general cannot be less than the ***machine precision***.

For small x, errors are damped by the function and hence are limited by the ***machine precision***.

For medium and large x, the error behaviour, like the function itself, is oscillatory, and hence only the absolute accuracy for the function can be maintained. For this range of x, the amplitude of the absolute error decays like $\sqrt{\frac{\pi x}{2}} e^{-x/\sqrt{2}}$ which implies a strong attenuation of error. Eventually, $\operatorname{ker} x$, which asymptotically behaves like $\sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}}$, becomes so small that it cannot be calculated without causing underflow, and the routine returns zero. Note that for large x the errors are dominated by those of the standard function exp.

8 Further Comments

Underflow may occur for a few values of x close to the zeros of $\ker x$, below the limit which causes a failure with IFAIL $= 1$.

9 Example

This example reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

9.1 Program Text

```
    Program s19acfe

!     S19ACF Example Program Text

!     Mark 24 Release. NAG Copyright 2012.

!     .. Use Statements ..
      Use nag_library, Only: nag_wp, s19acf
!     .. Implicit None Statement ..
      Implicit None
!     .. Parameters ..
      Integer, Parameter               :: nin = 5, nout = 6
!     .. Local Scalars ..
      Real (Kind=nag_wp)               :: x, y
      Integer                          :: ifail, ioerr
!     .. Executable Statements ..
      Write (nout,*) 'S19ACF Example Program Results'

!     Skip heading in data file
      Read (nin,*)

      Write (nout,*)
      Write (nout,*) '     X           Y'
      Write (nout,*)

data: Do
        Read (nin,*,Iostat=ioerr) x

        If (ioerr<0) Then
          Exit data
        End If

        ifail = -1
        y = s19acf(x,ifail)

        If (ifail<0) Then
          Exit data
        End If

        Write (nout,99999) x, y
      End Do data

99999 Format (1X,1P,2E12.3)
    End Program s19acfe
```

9.2 Program Data

```
S19ACF Example Program Data
             0.1
             1.0
             2.5
             5.0
             10.0
             15.0
```

9.3 Program Results

```
 S19ACF Example Program Results

      X           Y

    1.000E-01   2.420E+00
    1.000E+00   2.867E-01
    2.500E+00  -6.969E-02
    5.000E+00  -1.151E-02
    1.000E+01   1.295E-04
    1.500E+01  -1.514E-08
```