

# NAG Library Routine Document

## F01GAF

**Note:** before using this routine, please read the Users' Note for your implementation to check the interpretation of *bold italicised* terms and other implementation-dependent details.

### 1 Purpose

F01GAF computes the action of the matrix exponential  $e^{tA}$ , on the matrix  $B$ , where  $A$  is a real  $n$  by  $n$  matrix,  $B$  is a real  $n$  by  $m$  matrix and  $t$  is a real scalar.

### 2 Specification

```
SUBROUTINE F01GAF (N, M, A, LDA, B, LDB, T, IFAIL)
```

```
INTEGER N, M, LDA, LDB, IFAIL
```

```
REAL (KIND=nag_wp) A(LDA,*), B(LDB,*), T
```

### 3 Description

$e^{tA}B$  is computed using the algorithm described in Al-Mohy and Higham (2011) which uses a truncated Taylor series to compute the product  $e^{tA}B$  without explicitly forming  $e^{tA}$ .

### 4 References

Al-Mohy A H and Higham N J (2011) Computing the action of the matrix exponential, with an application to exponential integrators *SIAM J. Sci. Statist. Comput.* **33(2)** 488-511

Higham N J (2008) *Functions of Matrices: Theory and Computation* SIAM, Philadelphia, PA, USA

### 5 Parameters

- 1: N – INTEGER *Input*  
*On entry:*  $n$ , the order of the matrix  $A$ .  
*Constraint:*  $N \geq 0$ .
- 2: M – INTEGER *Input*  
*On entry:*  $m$ , the number of columns of the matrix  $B$ .  
*Constraint:*  $M \geq 0$ .
- 3: A(LDA,\*) – REAL (KIND=nag\_wp) array *Input/Output*  
**Note:** the second dimension of the array A must be at least N.  
*On entry:* the  $n$  by  $n$  matrix  $A$ .  
*On exit:*  $A$  is overwritten during the computation.
- 4: LDA – INTEGER *Input*  
*On entry:* the first dimension of the array A as declared in the (sub)program from which F01GAF is called.  
*Constraint:*  $LDA \geq \max(1, N)$ .

- 5: B(LDB,\*) – REAL (KIND=nag\_wp) array Input/Output  
**Note:** the second dimension of the array B must be at least M.  
*On entry:* the  $n$  by  $m$  matrix  $B$ .  
*On exit:* the  $n$  by  $m$  matrix  $e^{tA}B$ .
- 6: LDB – INTEGER Input  
*On entry:* the first dimension of the array B as declared in the (sub)program from which F01GAF is called.  
*Constraint:*  $LDB \geq \max(1, N)$ .
- 7: T – REAL (KIND=nag\_wp) Input  
*On entry:* the scalar  $t$ .
- 8: IFAIL – INTEGER Input/Output  
*On entry:* IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this parameter you should refer to Section 3.3 in the Essential Introduction for details.  
For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this parameter, the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**  
*On exit:* IFAIL = 0 unless the routine detects an error or a warning has been flagged (see Section 6).

## 6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

**Note:** this failure should not occur, and suggests that the routine has been called incorrectly. An unexpected internal error occurred when trying to balance the matrix  $A$ .

IFAIL = 2

An unexpected internal error has occurred. Please contact NAG.

IFAIL = -1

*On entry,*  $N = \langle value \rangle$ .  
*Constraint:*  $N \geq 0$ .

IFAIL = -2

*On entry,*  $M = \langle value \rangle$ .  
*Constraint:*  $M \geq 0$ .

IFAIL = -4

*On entry,*  $LDA = \langle value \rangle$  and  $N = \langle value \rangle$ .  
*Constraint:*  $LDA \geq \max(1, N)$ .

IFAIL = -6

On entry, LDB =  $\langle value \rangle$  and N =  $\langle value \rangle$ .  
Constraint: LDB  $\geq$  max(1,N).

IFAIL = -999

Allocation of memory failed.

## 7 Accuracy

For a symmetric matrix  $A$  (for which  $A^T = A$ ) the computed matrix  $e^{tA}B$  is guaranteed to be close to the exact matrix, that is, the method is forward stable. No such guarantee can be given for non-symmetric matrices. See Section 4 of Al-Mohy and Higham (2011) for details and further discussion.

## 8 Further Comments

The matrix  $e^{tA}B$  could be computed by explicitly forming  $e^{tA}$  using F01ECF and multiplying  $B$  by the result. However, experiments show that it is usually both more accurate and quicker to use F01GAF.

The cost of the algorithm is  $O(n^2m)$ . The precise cost depends on  $A$  since a combination of balancing, shifting and scaling is used prior to the Taylor series evaluation.

Approximately  $n^2 + (2m + 8)n$  of real allocatable memory is required by F01GAF.

F01HAF can be used to compute  $e^{tA}B$  for complex  $A$ ,  $B$ , and  $t$ . F01GBF provides an implementation of the algorithm with a reverse communication interface, which returns control to the user when matrix multiplications are required. This should be used if  $A$  is large and sparse.

## 9 Example

This example computes  $e^{tA}B$ , where

$$A = \begin{pmatrix} 0.7 & -0.2 & 1.0 & 0.3 \\ 0.3 & 0.7 & 1.2 & 1.0 \\ 0.9 & 0.0 & 0.2 & 0.7 \\ 2.4 & 0.1 & 0.0 & 0.2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.1 & 1.2 \\ 1.3 & 0.2 \\ 0.0 & 1.0 \\ 0.4 & -0.9 \end{pmatrix}$$

and

$$t = 1.2.$$

### 9.1 Program Text

```

Program f01gafe
!      F01GAF Example Program Text
!
!      Mark 24 Release. NAG Copyright 2012.
!
!      .. Use Statements ..
      Use nag_library, Only: f01gaf, nag_wp, x04caf
!      .. Implicit None Statement ..
      Implicit None
!      .. Parameters ..
      Integer, Parameter          :: nin = 5, nout = 6
!      .. Local Scalars ..

```

```

      Real (Kind=nag_wp)                :: t
      Integer                          :: i, ifail, lda, ldb, m, n
!    .. Local Arrays ..
      Real (Kind=nag_wp), Allocatable :: a(:,,:), b(:,,:)
!    .. Executable Statements ..
      Write (nout,*) 'F01GAF Example Program Results'
      Write (nout,*)
!    Skip heading in data file
      Read (nin,*)
      Read (nin,*) n, m, t

      lda = n
      ldb = n

      Allocate (a(lda,n))
      Allocate (b(ldb,m))

!    Read A from data file
      Read (nin,*)(a(i,1:n),i=1,n)

!    Read B from data file
      Read (nin,*)(b(i,1:m),i=1,n)

!    Find exp(tA)B
      ifail = 0
      Call f01gaf(n,m,a,lda,b,ldb,t,ifail)

      If (ifail==0) Then
!    Print solution
      ifail = 0
      Call x04caf('G','N',n,m,b,ldb,'exp(tA)B',ifail)
      End If

      End Program f01gafe

```

## 9.2 Program Data

F01GAF Example Program Data

```

4      2      1.2          :Values of N, M and T

0.7   -0.2   1.0   0.3
0.3   0.7   1.2   1.0
0.9   0.0   0.2   0.7
2.4   0.1   0.0   0.2 :End of matrix A

0.1   1.2
1.3   0.2
0.0   1.0
0.4   -0.9          :End of matrix B

```

## 9.3 Program Results

F01GAF Example Program Results

```

exp(tA)B
      1      2
1      0.2138      7.6756
2      4.9980      11.6051
3      0.8307      7.5468
4      1.2406      9.7261

```