NAG Library Routine Document

D05BYF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

D05BYF computes the fractional quadrature weights associated with the Backward Differentiation Formulae (BDF) of orders 4, 5 and 6. These weights can then be used in the solution of weakly singular equations of Abel type.

2 Specification

```
SUBROUTINE D05BYF (IORDER, IQ, LENFW, WT, SW, LDSW, WORK, LWK, IFAIL)

INTEGER             IORDER, IQ, LENFW, LDSW, LWK, IFAIL
REAL (KIND=nag_wp) WT(LENFW), SW(LDSW,2*IORDER-1), WORK(LWK)
```

3 Description

D05BYF computes the weights $W_{i,j}$ and ω_i for a family of quadrature rules related to a BDF method for approximating the integral:

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{\phi(s)}{\sqrt{t-s}}ds \simeq \sqrt{h}\sum_{j=0}^{2p-2} W_{i,j}\phi(j\times h) + \sqrt{h}\sum_{j=2p-1}^{i}\omega_{i-j}\phi(j\times h), \qquad 0\leq t\leq T, \tag{1}$$

with $t = i \times h (i \geq 0)$, for some given h. In (1), p is the order of the BDF method used and $W_{i,j}$, ω_i are the fractional starting and the fractional convolution weights respectively. The algorithm for the generation of ω_i is based on Newton's iteration. Fast Fourier transform (FFT) techniques are used for computing these weights and subsequently $W_{i,j}$ (see Baker and Derakhshan (1987) and Henrici (1979) for practical details and Lubich (1986) for theoretical details). Some special functions can be represented as the fractional integrals of simpler functions and fractional quadratures can be employed for their computation (see Lubich (1986)). A description of how these weights can be used in the solution of weakly singular equations of Abel type is given in Section 8.

4 References

Baker C T H and Derakhshan M S (1987) Computational approximations to some power series *Approximation Theory* (eds L Collatz, G Meinardus and G Nürnberger) **81** 11–20

Henrici P (1979) Fast Fourier methods in computational complex analysis *SIAM Rev.* **21** 481–529

Lubich Ch (1986) Discretized fractional calculus *SIAM J. Math. Anal.* **17** 704–719

5 Parameters

1: IORDER – INTEGER *Input*

On entry: p, the order of the BDF method to be used.

Constraint: $4 \leq$ IORDER ≤ 6.

2: IQ – INTEGER *Input*

On entry: determines the number of weights to be computed. By setting IQ to a value, $2^{\mathrm{IQ}+1}$ fractional convolution weights are computed.

Constraint: $\mathrm{IQ} \geq 0$.

3: LENFW – INTEGER *Input*

On entry: the dimension of the array WT as declared in the (sub)program from which D05BYF is called.

Constraint: $\mathrm{LENFW} \geq 2^{\mathrm{IQ}+2}$.

4: WT(LENFW) – REAL (KIND=nag_wp) array *Output*

On exit: the first $2^{\mathrm{IQ}+1}$ elements of WT contains the fractional convolution weights ω_i, for $i = 0, 1, \dots, 2^{\mathrm{IQ}+1} - 1$. The remainder of the array is used as workspace.

5: SW(LDSW,2 × IORDER − 1) – REAL (KIND=nag_wp) array *Output*

On exit: $\mathrm{SW}(i, j + 1)$ contains the fractional starting weights $W_{i-1,j}$, for $i = 1, 2, \dots, N$ and $j = 0, 1, \dots, 2 \times \mathrm{IORDER} - 2$, where $N = \left(2^{\mathrm{IQ}+1} + 2 \times \mathrm{IORDER} - 1\right)$.

6: LDSW – INTEGER *Input*

On entry: the first dimension of the array SW as declared in the (sub)program from which D05BYF is called.

Constraint: $\mathrm{LDSW} \geq 2^{\mathrm{IQ}+1} + 2 \times \mathrm{IORDER} - 1$.

7: WORK(LWK) – REAL (KIND=nag_wp) array *Workspace*
8: LWK – INTEGER *Input*

On entry: the dimension of the array WORK as declared in the (sub)program from which D05BYF is called.

Constraint: $\mathrm{LWK} \geq 2^{\mathrm{IQ}+3}$.

9: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this parameter you should refer to Section 3.3 in the Essential Introduction for details.

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this parameter, the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

On exit: $\mathrm{IFAIL} = 0$ unless the routine detects an error or a warning has been flagged (see Section 6).

6 Error Indicators and Warnings

If on entry $\mathrm{IFAIL} = 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

$\mathrm{IFAIL} = 1$

On entry, $\mathrm{IORDER} < 4$ or $\mathrm{IORDER} > 6$,
or $\mathrm{IQ} < 0$,
or $\mathrm{LENFW} < 2^{\mathrm{IQ}+2}$,

or LDSW $< 2^{\mathrm{IQ}+1} + 2 \times \mathrm{IORDER} - 1$,
or LWK $< 2^{\mathrm{IQ}+3}$.

7 Accuracy

Not applicable.

8 Further Comments

Fractional quadrature weights can be used for solving weakly singular integral equations of Abel type. In this section, we propose the following algorithm which you may find useful in solving a linear weakly singular integral equation of the form

$$y(t) = f(t) + \frac{1}{\sqrt{\pi}}\int_0^t \frac{K(t,s)y(s)}{\sqrt{t-s}}\,ds, \qquad 0 \le t \le T, \tag{2}$$

using D05BYF. In (2), $K(t,s)$ and $f(t)$ are given and the solution $y(t)$ is sought on a uniform mesh of size h such that $T = N \times h$. Discretization of (2) yields

$$y_i = f(i \times h) + \sqrt{h}\sum_{j=0}^{2p-2} W_{i,j}K(i \times h, j \times h)y_j + \sqrt{h}\sum_{j=2p-1}^{i} \omega_{i-j}K(i \times h, j \times h)y_j, \tag{3}$$

where $y_i \simeq y(i \times h)$, for $i = 1, 2, \ldots, N$. We propose the following algorithm for computing y_i from (3) after a call to D05BYF:

(a) Set $N = 2^{\mathrm{IQ}+1} + 2 \times \mathrm{IORDER} - 2$ and $h = T/N$.

(b) Equation (3) requires $2 \times \mathrm{IORDER} - 2$ starting values, y_j, for $j = 1, 2, \ldots, 2 \times \mathrm{IORDER} - 2$, with $y_0 = f(0)$. These starting values can be computed by solving the system

$$y_i = f(i \times h) + \sqrt{h}\sum_{j=0}^{2\times\mathrm{IORDER}-2} \mathrm{SW}(i+1, j+1)K(i \times h, j \times h)y_j, \qquad i = 1, 2, \ldots, 2 \times \mathrm{IORDER} - 2.$$

(c) Compute the inhomogeneous terms

$$\sigma_i = f(i \times h) + \sqrt{h}\sum_{j=0}^{2\times\mathrm{IORDER}-2} \mathrm{SW}(i+1, j+1)K(i \times h, j \times h)y_j, \qquad i = 2 \times \mathrm{IORDER} - 1, 2 \times \mathrm{IORDER}, \ldots, N.$$

(d) Start the iteration for $i = 2 \times \mathrm{IORDER} - 1, 2 \times \mathrm{IORDER}, \ldots, N$ to compute y_i from:

$$\left(1 - \sqrt{h}\mathrm{WT}(1)K(i \times h, i \times h)\right)y_i = \sigma_i + \sqrt{h}\sum_{j=2\times\mathrm{IORDER}-1}^{i-1} \mathrm{WT}(i-j+1)K(i \times h, j \times h)y_j.$$

Note that for nonlinear weakly singular equations, the solution of a nonlinear algebraic system is required at step (b) and a single nonlinear equation at step (d).

9 Example

The following example generates the first 16 fractional convolution and 23 fractional starting weights generated by the fourth-order BDF method.

9.1 Program Text

```
    Program d05byfe

!     D05BYF Example Program Text

!     Mark 24 Release. NAG Copyright 2012.

!     .. Use Statements ..
      Use nag_library, Only: d05byf, nag_wp
!     .. Implicit None Statement ..
```

```
      Implicit None
!     .. Parameters ..
      Integer, Parameter                :: iorder = 4, iq = 3
      Integer, Parameter                :: itiq = 2**(iq+1)
      Integer, Parameter                :: itpmt = 2*iorder - 1
      Integer, Parameter                :: ldsw = itiq + itpmt
      Integer, Parameter                :: lenfw = 2*itiq
      Integer, Parameter                :: lwk = 4*itiq
      Integer, Parameter                :: nout = 6
!     .. Local Scalars ..
      Integer                           :: i, ifail
!     .. Local Arrays ..
      Real (Kind=nag_wp)                :: sw(ldsw,itpmt), work(lwk), wt(lenfw)
!     .. Executable Statements ..
      Write (nout,*) 'D05BYF Example Program Results'

      ifail = 0
      Call d05byf(iorder,iq,lenfw,wt,sw,ldsw,work,lwk,ifail)

      Write (nout,*)
      Write (nout,*) 'Fractional convolution weights'
      Write (nout,*)

      Do i = 1, itiq
        Write (nout,99999) i - 1, wt(i)
      End Do

      Write (nout,*)
      Write (nout,*) 'Fractional starting weights'
      Write (nout,*)

      Do i = 1, ldsw
        Write (nout,99999) i - 1, sw(i,1:itpmt)
      End Do

99999 Format (1X,I5,7F9.4)
    End Program d05byfe
```

9.2 Program Data

None.

9.3 Program Results

```
 D05BYF Example Program Results

 Fractional convolution weights

     0   0.6928
     1   0.6651
     2   0.4589
     3   0.3175
     4   0.2622
     5   0.2451
     6   0.2323
     7   0.2164
     8   0.2006
     9   0.1878
    10   0.1780
    11   0.1700
    12   0.1629
    13   0.1566
    14   0.1508
    15   0.1457

 Fractional starting weights

     0   0.0000   0.0000   0.0000   0.0000   0.0000   0.0000   0.0000
     1   0.0565   2.8928  -6.7497  11.6491 -11.1355   5.5374  -1.1223
     2   0.0371   1.7401  -2.8628   6.5207  -6.4058   3.2249  -0.6583
```

```
   3   0.0300   1.3207  -2.4642   6.3612  -5.4478   2.7025  -0.5481
   4   0.0258   1.1217  -2.2620   5.3683  -3.7553   2.2132  -0.4549
   5   0.0230   0.9862  -2.0034   4.5005  -3.2772   2.7262  -0.4320
   6   0.0208   0.9001  -1.8989   4.2847  -3.5881   2.8201   0.2253
   7   0.0190   0.8506  -1.9250   4.4164  -4.0181   2.7932   0.1564
   8   0.0173   0.8177  -1.9697   4.5348  -4.2425   2.7458  -0.0697
   9   0.0160   0.7886  -1.9781   4.5318  -4.2769   2.6997  -0.2127
  10   0.0149   0.7603  -1.9548   4.4545  -4.2332   2.6541  -0.2620
  11   0.0140   0.7338  -1.9198   4.3619  -4.1782   2.6059  -0.2716
  12   0.0132   0.7097  -1.8842   4.2754  -4.1246   2.5544  -0.2767
  13   0.0125   0.6880  -1.8497   4.1933  -4.0662   2.5011  -0.2845
  14   0.0119   0.6681  -1.8153   4.1109  -4.0004   2.4479  -0.2915
  15   0.0114   0.6497  -1.7805   4.0279  -3.9304   2.3962  -0.2951
  16   0.0110   0.6327  -1.7461   3.9463  -3.8598   2.3466  -0.2958
  17   0.0105   0.6168  -1.7126   3.8677  -3.7907   2.2990  -0.2950
  18   0.0102   0.6020  -1.6804   3.7926  -3.7238   2.2536  -0.2935
  19   0.0098   0.5882  -1.6495   3.7209  -3.6589   2.2101  -0.2917
  20   0.0095   0.5752  -1.6199   3.6523  -3.5961   2.1686  -0.2895
  21   0.0093   0.5631  -1.5916   3.5867  -3.5356   2.1291  -0.2871
  22   0.0090   0.5517  -1.5644   3.5240  -3.4774   2.0914  -0.2844
```