NAG Library Function Document

nag_tsa_cp_pelt (g13nac)

1 Purpose

nag_tsa_cp_pelt (g13nac) detects change points in a univariate time series, that is, the time points at which some feature of the data, for example the mean, changes. Change points are detected using the PELT (Pruned Exact Linear Time) algorithm using one of a provided set of cost functions.

2 Specification

```
#include <nag.h>
#include <nagg13.h>
```

```
void nag_tsa_cp_pelt (Nag_TS_ChangeType ctype, Integer n, const double y[],
     double beta, Integer minss, const double param[], Integer *ntau,
     Integer tau[], double sparam[], NagError *fail)
```

3 Description

Let $y_{1:n} = \{y_j : j = 1, 2, \ldots, n\}$ denote a series of data and $\tau = \{\tau_i : i = 1, 2, \ldots, m\}$ denote a set of m ordered (strictly monotonic increasing) indices known as change points with $1 \le \tau_i \le n$ and $\tau_m = n$. For ease of notation we also define $\tau_0 = 0$. The m change points, τ, split the data into m segments, with the ith segment being of length n_i and containing $y_{\tau_{i-1}+1:\tau_i}$.

Given a cost function, $C(y_{\tau_{i-1}+1:\tau_i})$ nag_tsa_cp_pelt (g13nac) solves

$$\underset{m,\tau}{\text{minimize}} \sum_{i=1}^{m} \left(C(y_{\tau_{i-1}+1:\tau_i}) + \beta \right) \tag{1}$$

where β is a penalty term used to control the number of change points. This minimization is performed using the PELT algorithm of Killick *et al.* (2012). The PELT algorithm is guaranteed to return the optimal solution to (1) if there exists a constant K such that

$$C\left(y_{(u+1):v}\right) + C\left(y_{(v+1):w}\right) + K \le C\left(y_{(u+1):w}\right) \tag{2}$$

for all $u < v < w$.

nag_tsa_cp_pelt (g13nac) supplies four families of cost function. Each cost function assumes that the series, y, comes from some distribution, $D(\Theta)$. The parameter space, $\Theta = \{\theta, \phi\}$ is subdivided into θ containing those parameters allowed to differ in each segment and ϕ those parameters treated as constant across all segments. All four cost functions can then be described in terms of the likelihood function, L and are given by:

$$C\left(y_{(\tau_{i-1}+1):\tau_i}\right) = -2\log L\left(\hat{\theta}_i, \phi | y_{(\tau_{i-1}+1):\tau_i}\right)$$

where $\hat{\theta}_i$ is the maximum likelihood estimate of θ within the ith segment. In all four cases setting $K = 0$ satisfies equation (2). Four distributions are available: Normal, Gamma, Exponential and Poisson. Letting

$$S_i = \sum_{j=\tau_{i-1}}^{\tau_i} y_j$$

the log likelihoods and cost functions for the four distributions, and the available subdivisions of the parameter space are:

Normal distribution: $\Theta = \{\mu, \sigma^2\}$

$$-2\log L = \sum_{i=1}^{m} \sum_{j=\tau_{i-1}}^{\tau_i} \log(2\pi) + \log\left(\sigma_i^2\right) + \frac{\left(y_j - \mu_i\right)^2}{\sigma_i^2}$$

Mean changes: $\theta = \{\mu\}$

$$C(y_{\tau_{i-1}+1:\tau_i}) = \sum_{j=\tau_{i-1}}^{\tau_i} \frac{\left(y_j - n_i^{-1} S_i\right)^2}{\sigma^2}$$

Variance changes: $\theta = \{\sigma^2\}$

$$C(y_{\tau_{i-1}+1:\tau_i}) = n_i \left(\log \left(\sum_{j=\tau_{i-1}}^{\tau_i} \left(y_j - \mu\right)^2 \right) - \log n_i \right)$$

Both mean and variance change: $\theta = \{\mu, \sigma^2\}$

$$C(y_{\tau_{i-1}+1:\tau_i}) = n_i \left(\log \left(\sum_{j=\tau_{i-1}}^{\tau_i} \left(y_j - n_i^{-1} S_i\right)^2 \right) - \log n_i \right)$$

Gamma distribution: $\Theta = \{a, b\}$

$$-2\log L = 2 \times \sum_{i=1}^{m} \sum_{j=\tau_{i-1}}^{\tau_i} \log \Gamma(a_i) + a_i \log b_i + (1 - a_i) \log y_j + \frac{y_j}{b_i}$$

Scale changes: $\theta = \{b\}$

$$C(y_{\tau_{i-1}+1:\tau_i}) = 2 a n_i (\log S_i - \log(a n_i))$$

Exponential Distribution: $\Theta = \{\lambda\}$

$$-2\log L = 2 \times \sum_{i=1}^{m} \sum_{j=\tau_{i-1}}^{\tau_i} \log \lambda_i + \frac{y_j}{\lambda_i}$$

Mean changes: $\theta = \{\lambda\}$

$$C(y_{\tau_{i-1}+1:\tau_i}) = 2 n_i (\log S_i - \log n_i)$$

Poisson distribution: $\Theta = \{\lambda\}$

$$-2\log L = 2 \times \sum_{i=1}^{m} \sum_{j=\tau_{i-1}}^{\tau_i} \lambda_i - \operatorname{floor} y_j + 0.5 \log \lambda_i + \log \Gamma\left(\operatorname{floor} y_j + 0.5 + 1\right)$$

Mean changes: $\theta = \{\lambda\}$

$$C(y_{\tau_{i-1}+1:\tau_i}) = 2 S_i (\log n_i - \log S_i)$$

when calculating S_i for the Poisson distribution, the sum is calculated for $\operatorname{floor} y_i + 0.5$ rather than y_i.

4 References

Chen J and Gupta A K (2010) *Parameteric Statisical Change Point Analysis With Applications to Genetics Medicine and Finance* **Second Edition** Birkhäuser

Killick R, Fearnhead P and Eckely I A (2012) Optimal detection of changepoints with a linear computational cost *Journal of the American Statistical Association* **107:500** 1590–1598

5 Arguments

1: **ctype** – Nag_TS_ChangeType *Input*

On entry: a flag indicating the assumed distribution of the data and the type of change point being looked for.

ctype = Nag_NormalMean
Data from a Normal distribution, looking for changes in the mean, μ.

ctype = Nag_NormalStd
Data from a Normal distribution, looking for changes in the standard deviation σ.

ctype = Nag_NormalMeanStd
Data from a Normal distribution, looking for changes in the mean, μ and standard deviation σ.

ctype = Nag_GammaScale
Data from a Gamma distribution, looking for changes in the scale parameter b.

ctype = Nag_ExponentialLambda
Data from an exponential distribution, looking for changes in λ.

ctype = Nag_PoissonLambda
Data from a Poisson distribution, looking for changes in λ.

Constraint: **ctype** = Nag_NormalMean, Nag_NormalStd, Nag_NormalMeanStd, Nag_GammaScale, Nag_ExponentialLambda or Nag_PoissonLambda.

2: **n** – Integer *Input*

On entry: n, the length of the time series.

Constraint: $\mathbf{n} \geq 2$.

3: **y**[**n**] – const double *Input*

On entry: y, the time series.

if **ctype** = Nag_PoissonLambda, that is the data is assumed to come from a Poisson distribution, floor $y + 0.5$ is used in all calculations.

Constraints:

if **ctype** = Nag_GammaScale, Nag_ExponentialLambda or Nag_PoissonLambda, $\mathbf{y}[i-1] \geq 0$, for $i = 1, 2, \ldots, \mathbf{n}$;
if **ctype** = Nag_PoissonLambda, each value of **y** must be representable as an integer;
if **ctype** $\neq$ Nag_PoissonLambda, each value of **y** must be small enough such that $\mathbf{y}[i-1]^2$, for $i = 1, 2, \ldots, \mathbf{n}$, can be calculated without incurring overflow.

4: **beta** – double *Input*

On entry: β, the penalty term.

There are a number of standard ways of setting β, including:

SIC or BIC
$\beta = p \times \log(n)$

AIC
$\beta = 2p$

Hannan-Quinn
$\beta = 2p \times \log(\log(n))$

where p is the number of parameters being treated as estimated in each segment. This is usually set to 2 when **ctype** = Nag_NormalMeanStd and 1 otherwise.

If no penalty is required then set $\beta = 0$. Generally, the smaller the value of β the larger the number of suggested change points.

5: **minss** – Integer *Input*

On entry: the minimum distance between two change points, that is $\tau_i - \tau_{i-1} \geq$ **minss**.

Constraint: **minss** ≥ 2.

6: **param**[**1**] – const double *Input*

On entry: ϕ, values for the parameters that will be treated as fixed. If **ctype** $\neq$ Nag_GammaScale, **param** may be set to **NULL**.

If **ctype** = Nag_NormalMean

if **param** is **NULL**, σ, the standard deviation of the Normal distribution, is estimated from the full input data. Otherwise $\sigma =$ **param**[0].

If **ctype** = Nag_NormalStd

If **param** is **NULL**, μ, the mean of the Normal distribution, is estimated from the full input data. Otherwise $\mu =$ **param**[0].

If **ctype** = Nag_GammaScale, **param**[0] must hold the shape, a, for the Gamma distribution, otherwise **param** is not referenced.

Constraint: if **ctype** = Nag_NormalMean or Nag_GammaScale, **param**[0] > 0.0.

7: **ntau** – Integer * *Output*

On exit: m, the number of change points detected.

8: **tau**[**n**] – Integer *Output*

On exit: the first m elements of **tau** hold the location of the change points. The ith segment is defined by $y_{(\tau_{i-1}+1)}$ to y_{τ_i}, where $\tau_0 = 0$ and $\tau_i =$ **tau**$[i-1], 1 \leq i \leq m$.

The remainder of **tau** is used as workspace.

9: **sparam**[$\mathbf{2} \times \mathbf{n} + \mathbf{2}$] – double *Output*

On exit: the estimated values of the distribution parameters in each segment

ctype = Nag_NormalMean, Nag_NormalStd or Nag_NormalMeanStd
sparam$[2i-2] = \mu_i$ and **sparam**$[2i-1] = \sigma_i$ for $i = 1, 2, \ldots, m$, where μ_i and σ_i is the mean and standard deviation, respectively, of the values of y in the ith segment.

It should be noted that $\sigma_i = \sigma_j$ when **ctype** = Nag_NormalMean and $\mu_i = \mu_j$ when **ctype** = Nag_NormalStd, for all i and j.

ctype = Nag_GammaScale
sparam$[2i-2] = a_i$ and **sparam**$[2i-1] = b_i$ for $i = 1, 2, \ldots, m$, where a_i and b_i are the shape and scale parameters, respectively, for the values of y in the ith segment. It should be noted that $a_i =$ **param**[0] for all i.

ctype = Nag_ExponentialLambda or Nag_PoissonLambda
sparam$[i-1] = \lambda_i$ for $i = 1, 2, \ldots, m$, where λ_i is the mean of the values of y in the ith segment.

The remainder of **sparam** is used as workspace.

10: **fail** – NagError * *Input/Output*

The NAG error argument (see Section 3.6 in the Essential Introduction).

6 Error Indicators and Warnings

NE_ALLOC_FAIL

Dynamic memory allocation failed.
See Section 3.2.1.2 in the Essential Introduction for further information.

NE_BAD_PARAM

On entry, argument ⟨*value*⟩ had an illegal value.

NE_INT

On entry, **minss** $=$ ⟨*value*⟩.
Constraint: **minss** ≥ 2.

On entry, **n** $=$ ⟨*value*⟩.
Constraint: **n** ≥ 2.

NE_INTERNAL_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

An unexpected error has been triggered by this function. Please contact NAG.
See Section 3.6.6 in the Essential Introduction for further information.

NE_NO_LICENCE

Your licence key may have expired or may not have been installed correctly.
See Section 3.6.5 in the Essential Introduction for further information.

NE_REAL

On entry, **ctype** $=$ ⟨*value*⟩ and **param**[0] $=$ ⟨*value*⟩.
Constraint: if **ctype** $=$ Nag_NormalMean or Nag_GammaScale and **param** is not **NULL**, then **param**[0] > 0.0.

NE_REAL_ARRAY

On entry, **ctype** $=$ ⟨*value*⟩ and **y**[⟨*value*⟩] $=$ ⟨*value*⟩.
Constraint: if **ctype** $=$ Nag_GammaScale, Nag_ExponentialLambda or Nag_PoissonLambda then **y**$[i-1] \geq 0.0$, for $i = 1, 2, \ldots,$ **n**.

On entry, **y**[⟨*value*⟩] $=$ ⟨*value*⟩, is too large.

NW_TRUNCATED

To avoid overflow some truncation occurred when calculating the cost function, C. All output is returned as normal.

To avoid overflow some truncation occurred when calculating the parameter estimates returned in **sparam**. All output is returned as normal.

7 Accuracy

For efficiency reasons, when calculating the cost functions, C and the parameter estimates returned in **sparam**, this function makes use of the mathematical identities:

$$\sum_{j=\mathrm{u}}^{v} {y_j}^2 = \sum_{j=1}^{v} {y_j}^2 - \sum_{j=1}^{u-1} {y_j}^2$$

and

$$\sum_{j=1}^{n} (y_j - \bar{y})^2 = \left(\sum_{j=1}^{n} y_j^2\right) - n\bar{y}^2$$

where $\bar{y} = n^{-1}\sum_{j=1}^{n} y_j$.

The input data, y, is scaled in order to try and mitigate some of the known instabilities associated with using these formulations. The results returned by nag_tsa_cp_pelt (g13nac) should be sufficient for the majority of datasets. If a more stable method of calculating C is deemed necessary, nag_tsa_cp_pelt_user (g13nbc) can be used and the method chosen implemented in the user-supplied cost function.

8 Parallelism and Performance

Not applicable.

9 Further Comments

None.

10 Example

This example identifies changes in the mean, under the assumption that the data is normally distributed, for a simulated dataset with 100 observations. A BIC penalty is used, that is $\beta = \log n \approx 4.6$, the minimum segment size is set to 2 and the variance is fixed at 1 across the whole input series.

10.1 Program Text

```
/* nag_tsa_cp_pelt (g13nac) Example Program.
 *
 * Copyright 2014 Numerical Algorithms Group.
 *
 * Mark 25, 2014.
 */
/* Pre-processor includes */
#include <stdio.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nagg13.h>

int main(void)
{
  /* Integer scalar and array declarations */
  Integer i, minss, n, ntau;
  Integer exit_status = 0;
  Integer *tau = 0;

  /* NAG structures and types */
  NagError fail;
  Nag_TS_ChangeType ctype;
  Nag_Boolean param_supplied;

  /* Double scalar and array declarations */
  double beta;
  double *param=0, *sparam = 0, *y = 0;

  /* Character scalar and array declarations */
  char cctype[40], cparam_supplied[40];

  /* Initialise the error structure */
  INIT_FAIL(fail);

  printf("nag_tsa_cp_pelt (g13nac) Example Program Results\n\n");

  /* Skip heading in data file */
#ifdef _WIN32
  scanf_s("%*[^\n] ");
#else
  scanf("%*[^\n] ");
#endif

  /* Read in the problem size */
#ifdef _WIN32
  scanf_s("%"NAG_IFMT"%*[^\n] ",&n);
#else
  scanf("%"NAG_IFMT"%*[^\n] ",&n);
#endif

  /* Allocate memory to hold the input series */
  if (!(y = NAG_ALLOC(n, double)))
    {
      printf("Allocation failure\n");
      exit_status = -1;
      goto END;
    }

  /* Read in the input series */
  for (i = 0; i < n; i++)
    {
#ifdef _WIN32
      scanf_s("%lf",&y[i]);
#else
      scanf("%lf",&y[i]);
#endif
    }
#ifdef _WIN32
  scanf_s("%*[^\n] ");
#else
  scanf("%*[^\n] ");
#endif

  /* Read in the type of change point, penalty and minimum segment size */
#ifdef _WIN32
  scanf_s("%39s%39s%lf%"NAG_IFMT"%*[^\n] ", cctype, _countof(cctype),
          cparam_supplied, _countof(cparam_supplied), &beta, &minss);
#else
   scanf("%39s%39s%lf%"NAG_IFMT"%*[^\n]  ", cctype, cparam_supplied, &beta,
&minss);
#endif
  ctype = (Nag_TS_ChangeType) nag_enum_name_to_value(cctype);
  param_supplied = (Nag_Boolean) nag_enum_name_to_value(cparam_supplied);

  /* Read in the distribution parameter (if required) */
  if (param_supplied)
    {
      if (!(param = NAG_ALLOC(1, double)))
        {
          printf("Allocation failure\n");
          exit_status = -1;
          goto END;
        }
#ifdef _WIN32
      scanf_s("%lf",&param[0]);
#else
```

```
      scanf("%lf",&param[0]);
#endif
#ifdef _WIN32
      scanf_s("%*[^\n] ");
#else
      scanf("%*[^\n] ");
#endif
    }

  /* Allocate output arrays */
  if (!(tau = NAG_ALLOC(n, Integer)) ||
      !(sparam = NAG_ALLOC(2*n+2,double)))
    {
      printf("Allocation failure\n");
      exit_status = -1;
      goto END;
    }

  /* Call nag_tsa_cp_pelt (g13nac) to to detect change points */
  nag_tsa_cp_pelt(ctype,n,y,beta,minss,param,&ntau,tau,sparam,&fail);
  if (fail.code != NE_NOERROR)
    {
      if (fail.code != NW_TRUNCATED) {
        printf("Error from nag_tsa_cp_pelt (g13nac).\n%s\n",
               fail.message);
        exit_status = 1;
        goto END;
      }
    }

  /* Display the results */
  if (ctype == Nag_ExponentialLambda || ctype == Nag_PoissonLambda)
    {
      /* Exponential or Poisson distribution */
      printf("  -- Change Points --      Distribution\n");
      printf("   Number     Position      Parameter\n");
      printf(" ====================================\n");
      for (i = 0; i < ntau; i++)
        {
          printf(" %4"NAG_IFMT"      %6"NAG_IFMT"    %12.2f\n",
                 i+1,tau[i],sparam[i]);
        }
    }
  else
    {
      /* Normal or Gamma distribution */
      printf("  -- Change Points --         --- Distribution ---\n");
      printf("  Number     Position              Parameters\n");
      printf(" ==================================================\n");
      for (i = 0; i < ntau; i++)
        {
          printf(" %4"NAG_IFMT"       %6"NAG_IFMT"    %12.2f    %12.2f\n",
                 i+1, tau[i], sparam[2*i], sparam[2*i+1]);
        }
    }
  if (fail.code == NW_TRUNCATED)
    {
      printf("Some truncation occurred internally to avoid overflow.\n");
    }

 END:
  NAG_FREE(y);
  NAG_FREE(param);
  NAG_FREE(tau);
  NAG_FREE(sparam);

  return(exit_status);
}
```

10.2 Program Data

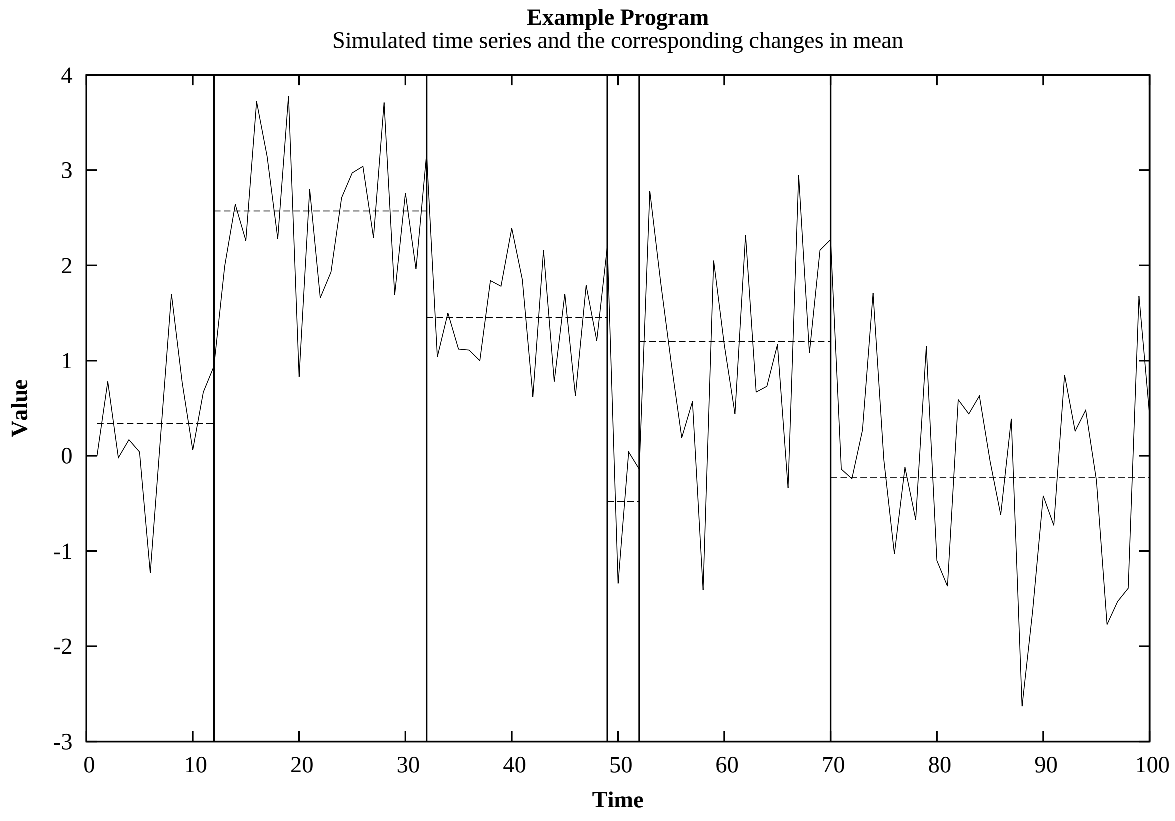
```
nag_tsa_cp_pelt (g13nac) Example Program Data
100                               :: n
 0.00  0.78 -0.02  0.17  0.04 -1.23  0.24  1.70  0.77  0.06
 0.67  0.94  1.99  2.64  2.26  3.72  3.14  2.28  3.78  0.83
 2.80  1.66  1.93  2.71  2.97  3.04  2.29  3.71  1.69  2.76
 1.96  3.17  1.04  1.50  1.12  1.11  1.00  1.84  1.78  2.39
 1.85  0.62  2.16  0.78  1.70  0.63  1.79  1.21  2.20 -1.34
 0.04 -0.14  2.78  1.83  0.98  0.19  0.57 -1.41  2.05  1.17
 0.44  2.32  0.67  0.73  1.17 -0.34  2.95  1.08  2.16  2.27
-0.14 -0.24  0.27  1.71 -0.04 -1.03 -0.12 -0.67  1.15 -1.10
-1.37  0.59  0.44  0.63 -0.06 -0.62  0.39 -2.63 -1.63 -0.42
-0.73  0.85  0.26  0.48 -0.26 -1.77 -1.53 -1.39  1.68  0.43 :: End of y
Nag_NormalMean  Nag_TRUE  4.6  2  :: ctype,param_supplied,beta,minss
1.0                               :: param[0]
```

10.3 Program Results

```
nag_tsa_cp_pelt (g13nac) Example Program Results

  -- Change Points --         --- Distribution ---
  Number     Position              Parameters
 ===================================================
     1           12             0.34            1.00
     2           32             2.57            1.00
     3           49             1.45            1.00
     4           52            -0.48            1.00
     5           70             1.20            1.00
     6          100            -0.23            1.00
```

This example plot shows the original data series, the estimated change points and the estimated mean in each of the identified segments.


Example Program
Simulated time series and the corresponding changes in mean
Value
Time