

## NAG Library Function Document

### nag\_herm\_posdef\_band\_lin\_solve (f04cfc)

#### 1 Purpose

nag\_herm\_posdef\_band\_lin\_solve (f04cfc) computes the solution to a complex system of linear equations  $AX = B$ , where  $A$  is an  $n$  by  $n$  Hermitian positive definite band matrix of band width  $2k + 1$ , and  $X$  and  $B$  are  $n$  by  $r$  matrices. An estimate of the condition number of  $A$  and an error bound for the computed solution are also returned.

#### 2 Specification

```
#include <nag.h>
#include <nagf04.h>

void nag_herm_posdef_band_lin_solve (Nag_OrderType order, Nag_UploType uplo,
    Integer n, Integer kd, Integer nrhs, Complex ab[], Integer pdab,
    Complex b[], Integer pdb, double *rcond, double *errbnd, NagError *fail)
```

#### 3 Description

The Cholesky factorization is used to factor  $A$  as  $A = U^H U$ , if **uplo** = Nag\_Upper, or  $A = LL^H$ , if **uplo** = Nag\_Lower, where  $U$  is an upper triangular band matrix with  $k$  superdiagonals, and  $L$  is a lower triangular band matrix with  $k$  subdiagonals. The factored form of  $A$  is then used to solve the system of equations  $AX = B$ .

#### 4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia <http://www.netlib.org/lapack/lug>

Higham N J (2002) *Accuracy and Stability of Numerical Algorithms* (2nd Edition) SIAM, Philadelphia

#### 5 Arguments

- 1: **order** – Nag\_OrderType *Input*  
*On entry:* the **order** argument specifies the two-dimensional storage scheme being used, i.e., row-major ordering or column-major ordering. C language defined storage is specified by **order** = Nag\_RowMajor. See Section 3.2.1.3 in the Essential Introduction for a more detailed explanation of the use of this argument.  
*Constraint:* **order** = Nag\_RowMajor or Nag\_ColMajor.
- 2: **uplo** – Nag\_UploType *Input*  
*On entry:* if **uplo** = Nag\_Upper, the upper triangle of the matrix  $A$  is stored.  
 If **uplo** = Nag\_Lower, the lower triangle of the matrix  $A$  is stored.  
*Constraint:* **uplo** = Nag\_Upper or Nag\_Lower.
- 3: **n** – Integer *Input*  
*On entry:* the number of linear equations  $n$ , i.e., the order of the matrix  $A$ .  
*Constraint:* **n**  $\geq$  0.

- 4: **kd** – Integer *Input*  
*On entry:* the number of superdiagonals  $k$  (and the number of subdiagonals) of the band matrix  $A$ .  
*Constraint:*  $\mathbf{kd} \geq 0$ .
- 5: **nrhs** – Integer *Input*  
*On entry:* the number of right-hand sides  $r$ , i.e., the number of columns of the matrix  $B$ .  
*Constraint:*  $\mathbf{nrhs} \geq 0$ .
- 6: **ab**[*dim*] – Complex *Input/Output*  
**Note:** the dimension, *dim*, of the array **ab** must be at least  $\max(1, \mathbf{pdab} \times \mathbf{n})$ .  
*On entry:*  
     if **uplo** = Nag\_Upper then  
         if **order** = Nag\_ColMajor,  $a_{ij}$  is stored in  $\mathbf{ab}[(j-1) \times \mathbf{pdab} + \mathbf{kd} + i - j]$ ;  
         if **order** = Nag\_RowMajor,  $a_{ij}$  is stored in  $\mathbf{ab}[(i-1) \times \mathbf{pdab} + j - i]$ .  
     for  $\max(1, j - \mathbf{kd}) \leq i \leq j$ ;  
     if **uplo** = Nag\_Lower then  
         if **order** = Nag\_ColMajor,  $a_{ij}$  is stored in  $\mathbf{ab}[(j-1) \times \mathbf{pdab} + i - j]$ ;  
         if **order** = Nag\_RowMajor,  $a_{ij}$  is stored in  $\mathbf{ab}[(i-1) \times \mathbf{pdab} + \mathbf{kd} + j - i]$ .  
     for  $j \leq i \leq \min(n, j + \mathbf{kd})$ ,  
 where  $\mathbf{pdab} \geq \mathbf{kd} + 1$  is the stride separating diagonal matrix elements in the array **ab**.  
 See Section 9 below for further details.  
*On exit:* if **fail.code** = NE\_NOERROR or NE\_RCOND, the factor  $U$  or  $L$  from the Cholesky factorization  $A = U^H U$  or  $A = LL^H$ , in the same storage format as  $A$ .
- 7: **pdab** – Integer *Input*  
*On entry:* the stride separating row or column elements (depending on the value of **order**) of the matrix  $A$  in the array **ab**.  
*Constraint:*  $\mathbf{pdab} \geq \mathbf{kd} + 1$ .
- 8: **b**[*dim*] – Complex *Input/Output*  
**Note:** the dimension, *dim*, of the array **b** must be at least  
 $\max(1, \mathbf{pdb} \times \mathbf{nrhs})$  when **order** = Nag\_ColMajor;  
 $\max(1, \mathbf{n} \times \mathbf{pdb})$  when **order** = Nag\_RowMajor.  
 The  $(i, j)$ th element of the matrix  $B$  is stored in  
 $\mathbf{b}[(j-1) \times \mathbf{pdb} + i - 1]$  when **order** = Nag\_ColMajor;  
 $\mathbf{b}[(i-1) \times \mathbf{pdb} + j - 1]$  when **order** = Nag\_RowMajor.  
*On entry:* the  $n$  by  $r$  matrix of right-hand sides  $B$ .  
*On exit:* if **fail.code** = NE\_NOERROR or NE\_RCOND, the  $n$  by  $r$  solution matrix  $X$ .
- 9: **pdb** – Integer *Input*  
*On entry:* the stride separating row or column elements (depending on the value of **order**) in the array **b**.

Constraints:

if **order** = Nag\_ColMajor, **pdb**  $\geq \max(1, \mathbf{n})$ ;  
 if **order** = Nag\_RowMajor, **pdb**  $\geq \max(1, \mathbf{nrhs})$ .

10: **rcond** – double \* Output

On exit: if **fail.code** = NE\_NOERROR or NE\_RCOND, an estimate of the reciprocal of the condition number of the matrix  $A$ , computed as  $\mathbf{rcond} = 1 / (\|A\|_1 \|A^{-1}\|_1)$ .

11: **errbnd** – double \* Output

On exit: if **fail.code** = NE\_NOERROR or NE\_RCOND, an estimate of the forward error bound for a computed solution  $\hat{x}$ , such that  $\|\hat{x} - x\|_1 / \|x\|_1 \leq \mathbf{errbnd}$ , where  $\hat{x}$  is a column of the computed solution returned in the array **b** and  $x$  is the corresponding column of the exact solution  $X$ . If **rcond** is less than *machine precision*, then **errbnd** is returned as unity.

12: **fail** – NagError \* Input/Output

The NAG error argument (see Section 3.6 in the Essential Introduction).

## 6 Error Indicators and Warnings

### NE\_ALLOC\_FAIL

Dynamic memory allocation failed.

### NE\_BAD\_PARAM

On entry, argument  $\langle value \rangle$  had an illegal value.

### NE\_INT

On entry, **kd** =  $\langle value \rangle$ .

Constraint: **kd**  $\geq 0$ .

On entry, **n** =  $\langle value \rangle$ .

Constraint: **n**  $\geq 0$ .

On entry, **nrhs** =  $\langle value \rangle$ .

Constraint: **nrhs**  $\geq 0$ .

On entry, **pdab** =  $\langle value \rangle$ .

Constraint: **pdab**  $> 0$ .

On entry, **pdb** =  $\langle value \rangle$ .

Constraint: **pdb**  $> 0$ .

### NE\_INT\_2

On entry, **pdab** =  $\langle value \rangle$  and **kd** =  $\langle value \rangle$ .

Constraint: **pdab**  $\geq \mathbf{kd} + 1$ .

On entry, **pdb** =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .

Constraint: **pdb**  $\geq \max(1, \mathbf{n})$ .

### NE\_INTERNAL\_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

**NE\_POS\_DEF**

The principal minor of order  $\langle value \rangle$  of the matrix  $A$  is not positive definite. The factorization has not been completed and the solution could not be computed.

**NE\_RCOND**

A solution has been computed, but **rcond** is less than *machine precision* so that the matrix  $A$  is numerically singular.

**7 Accuracy**

The computed solution for a single right-hand side,  $\hat{x}$ , satisfies an equation of the form

$$(A + E)\hat{x} = b,$$

where

$$\|E\|_1 = O(\epsilon)\|A\|_1$$

and  $\epsilon$  is the *machine precision*. An approximate error bound for the computed solution is given by

$$\frac{\|\hat{x} - x\|_1}{\|x\|_1} \leq \kappa(A) \frac{\|E\|_1}{\|A\|_1},$$

where  $\kappa(A) = \|A^{-1}\|_1 \|A\|_1$ , the condition number of  $A$  with respect to the solution of the linear equations. `nag_herm_posdef_band_lin_solve` (f04cfc) uses the approximation  $\|E\|_1 = \epsilon \|A\|_1$  to estimate **errbnd**. See Section 4.4 of Anderson *et al.* (1999) for further details.

**8 Parallelism and Performance**

`nag_herm_posdef_band_lin_solve` (f04cfc) is threaded by NAG for parallel execution in multithreaded implementations of the NAG Library.

`nag_herm_posdef_band_lin_solve` (f04cfc) makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the Users' Note for your implementation for any additional implementation-specific information.

**9 Further Comments**

The band storage schemes for the array **ab** are identical to the storage schemes for symmetric and Hermitian band matrices in Chapter f07. See Section 3.3.4 in the f07 Chapter Introduction for details of the storage schemes and an illustrated example.

If **uplo** = Nag\_Upper then the elements of the stored upper triangular part of  $A$  are overwritten by the corresponding elements of the upper triangular matrix  $U$ . Similarly, if **uplo** = Nag\_Lower then the elements of the stored lower triangular part of  $A$  are overwritten by the corresponding elements of the lower triangular matrix  $L$ .

Assuming that  $n \gg k$ , the total number of floating-point operations required to solve the equations  $AX = B$  is approximately  $n(k+1)^2$  for the factorization and  $4nkr$  for the solution following the factorization. The condition number estimation typically requires between four and five solves and never more than eleven solves, following the factorization.

In practice the condition number estimator is very reliable, but it can underestimate the true condition number; see Section 15.3 of Higham (2002) for further details.

The real analogue of `nag_herm_posdef_band_lin_solve` (f04cfc) is `nag_real_sym_posdef_band_lin_solve` (f04bfc).

## 10 Example

This example solves the equations

$$AX = B,$$

where  $A$  is the Hermitian positive definite band matrix

$$A = \begin{pmatrix} 9.39 & 1.08 - 1.73i & 0 & 0 \\ 1.08 + 1.73i & 1.69 & -0.04 + 0.29i & 0 \\ 0 & -0.04 - 0.29i & 2.65 & -0.33 + 2.24i \\ 0 & 0 & -0.33 - 2.24i & 2.17 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -12.42 + 68.42i & 54.30 - 56.56i \\ -9.93 + 0.88i & 18.32 + 4.76i \\ -27.30 - 0.01i & -4.40 + 9.97i \\ 5.31 + 23.63i & 9.43 + 1.41i \end{pmatrix}.$$

An estimate of the condition number of  $A$  and an approximate error bound for the computed solutions are also printed.

### 10.1 Program Text

```

/* nag_herm_posdef_band_lin_solve (f04cfc) Example Program.
 *
 * Copyright 2004 Numerical Algorithms Group.
 *
 * Mark 8, 2004.
 */

#include <stdio.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nagf04.h>
#include <nagx04.h>

int main(void)
{
    double          errbnd, rcond;
    Integer         exit_status, i, j, kd, n, nrhs, pdab, pdb;

    /* Arrays */
    char            nag_enum_arg[20];
    char            *clabs = 0, *rlabs = 0;
    Complex         *ab = 0, *b = 0;

    /* Nag types */
    NagError        fail;
    Nag_OrderType   order;
    Nag_UploType    uplo;

#ifdef NAG_COLUMN_MAJOR
#define AB_U(I, J) ab[(J-1)*pdab + kd + I - J]
#define AB_L(I, J) ab[(J-1)*pdab + I - J]
#define B(I, J)    b[(J-1)*pdb + I - 1]
    order = Nag_ColMajor;
#else
#define AB_U(I, J) ab[(I-1)*pdab + J - I]
#define AB_L(I, J) ab[(I-1)*pdab + kd + J - I]
#define B(I, J)    b[(I-1)*pdb + J - 1]
    order = Nag_RowMajor;
#endif

    exit_status = 0;
    INIT_FAIL(fail);

```

```

printf("nag_herm_posdef_band_lin_solve (f04cfc)"
      " Example Program Results\n\n");

/* Skip heading in data file */
scanf("%*[\n] ");
scanf("%ld%ld%ld%*[\n] ", &n, &kd, &nrhs);
if (n > 0 && kd > 0 && nrhs > 0)
{
    /* Allocate memory */
    if (!(clabs = NAG_ALLOC(2, char)) ||
        !(rlabs = NAG_ALLOC(2, char)) ||
        !(ab = NAG_ALLOC((kd+1)*n, Complex)) ||
        !(b = NAG_ALLOC(n*nrhs, Complex)))
    {
        printf("Allocation failure\n");
        exit_status = -1;
        goto END;
    }
    pdab = kd+1;
#ifdef NAG_COLUMN_MAJOR
    pdb = n;
#else
    pdb = nrhs;
#endif
}
else
{
    printf("%s\n", "One or more of n, kd and nrhs is too small");
    exit_status = 1;
    return exit_status;
}

/* Read uplo storage name for the matrix A and convert to value. */
scanf("%19s%*[\n] ", nag_enum_arg);

/* nag_enum_name_to_value (x04nac).
 * Converts NAG enum member name to value
 */
uplo = (Nag_UploType) nag_enum_name_to_value(nag_enum_arg);

/* Read the upper or lower triangular part of the band matrix A */
/* from data file */

if (uplo == Nag_Upper)
{
    for (i = 1; i <= n; ++i)
    {
        for (j = i; j <= MIN(n, i + kd); ++j)
        {
            scanf(" ( %lf , %lf )", &AB_U(i, j).re, &AB_U(i, j).im);
        }
        scanf("%*[\n] ");
    }
}
else
{
    for (i = 1; i <= n; ++i)
    {
        for (j = MAX(1, i - kd); j <= i; ++j)
        {
            scanf(" ( %lf , %lf )", &AB_L(i, j).re, &AB_L(i, j).im);
        }
        scanf("%*[\n] ");
    }
}

/* Read B from data file */
for (i = 1; i <= n; ++i)
{
    for (j = 1; j <= nrhs; ++j)
    {

```

```

        scanf(" ( %lf , %lf )", &B(i, j).re, &B(i, j).im);
    }
}
scanf("%*[\n] ");

/* Solve the equations AX = B for X */
/* nag_herm_posdef_band_lin_solve (f04cfc).
 * Computes the solution and error-bound to a complex
 * Hermitian positive-definite banded system of linear
 * equations
 */
nag_herm_posdef_band_lin_solve(order, uplo, n, kd, nrhs, ab, pdab, b,
                               pdb, &rcond, &errbnd, &fail);

if (fail.code == NE_NOERROR)
{
    /* Print solution, estimate of condition number and approximate *
     * error bound */
    /* nag_gen_complx_mat_print_comp (x04dbc).
     * Print complex general matrix (comprehensive)
     */
    fflush(stdout);
    nag_gen_complx_mat_print_comp(order, Nag_GeneralMatrix,
                                  Nag_NonUnitDiag, n, nrhs, b, pdb,
                                  Nag_BracketForm, "%7.4f",
                                  "Solution", Nag_IntegerLabels, 0,
                                  Nag_IntegerLabels, 0, 80, 0, 0,
                                  &fail);

    if (fail.code != NE_NOERROR)
    {
        printf(
            "Error from nag_gen_complx_mat_print_comp (x04dbc).\n%s\n",
            fail.message);
        exit_status = 1;
        goto END;
    }

    printf("\n%s\n%4s%10.1e\n\n", "Estimate of condition number", "",
           1.0/rcond);
    printf("\n%s\n%4s%10.1e\n\n",
           "Estimate of error bound for computed solutions", "",
           errbnd);
}
else if (fail.code == NE_RCOND)
{
    /* Matrix A is numerically singular. Print estimate of */
    /* reciprocal of condition number and solution. */

    printf("\n");
    printf("%s\n%4s%10.1e\n\n",
           "Estimate of reciprocal of condition number", "", rcond);
    /* nag_gen_complx_mat_print_comp (x04dbc), see above. */
    fflush(stdout);
    nag_gen_complx_mat_print_comp(order, Nag_GeneralMatrix,
                                  Nag_NonUnitDiag, n, nrhs, b, pdb,
                                  Nag_BracketForm, "%7.4f",
                                  "Solution", Nag_IntegerLabels, 0,
                                  Nag_IntegerLabels, 0, 80, 0, 0,
                                  &fail);

    if (fail.code != NE_NOERROR)
    {
        printf(
            "Error from nag_gen_complx_mat_print_comp (x04dbc).\n%s\n",
            fail.message);
        exit_status = 1;
        goto END;
    }
}
else if (fail.code == NE_POS_DEF)
{
    /* The matrix A is not positive definite to working precision */

```

```

        printf("%s%3ld%s\n\n", "The leading minor of order ",
               fail.errnum, " is not positive definite");
    }
else
    {
        printf(
            "Error from nag_herm_posdef_band_lin_solve (f04cfc).\n%s\n",
            fail.message);
        exit_status = 1;
        goto END;
    }

END:
    NAG_FREE(clabs);
    NAG_FREE(rlabs);
    NAG_FREE(ab);
    NAG_FREE(b);

    return exit_status;
}
#undef AB
#undef B

```

## 10.2 Program Data

nag\_herm\_posdef\_band\_lin\_solve (f04cfc) Example Program Data

```

    4           1           2           :Values of N, KD and NRHS
    Nag_Upper
    ( 9.39, 0.00) ( 1.08, -1.73)
                   ( 1.69, 0.00) ( -0.04, 0.29)
                                   ( 2.65, 0.00) ( -0.33, 2.24)
                                           ( 2.17, 0.00) :End of A

    (-12.42, 68.42) ( 54.30,-56.56)
    (-9.93, 0.88) ( 18.32, 4.76)
    (-27.30, -0.01) ( -4.40, 9.97)
    ( 5.31, 23.63) ( 9.43, 1.41)           :End of B

```

## 10.3 Program Results

nag\_herm\_posdef\_band\_lin\_solve (f04cfc) Example Program Results

```

Solution
           1           2
1 (-1.0000, 8.0000) ( 5.0000,-6.0000)
2 ( 2.0000,-3.0000) ( 2.0000, 3.0000)
3 (-4.0000,-5.0000) (-8.0000, 4.0000)
4 ( 7.0000, 6.0000) (-1.0000,-7.0000)

Estimate of condition number
    1.3e+02

Estimate of error bound for computed solutions
    1.5e-14

```

---