Index-tracking portfolio optimization model

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Abstract

In the present tutorial report we examine the theory and computational aspects behind the index-tracking portfolio optimization model. This model is compared with Markowitz mean-variance model. This report is distributed with an example in C using the NAG C Library.

1 Introduction

Index-tracking is a form of passive fund management. The index-tracking problem is the problem of reproducing the performance of a stock market index by considering a portfolio of assets comprised on the index. This approach differs from the *full replication* strategy, where a fund purchases all the stocks that make up a particular market index. A passively managed fund whose objective is to reproduce the return on an index is known as an index fund, or a tracker fund. The classical index-tracking approach represents the problem in a least square framework with errors computed using a sample of historical data. A new approach described in [1], which this report is based on, replaces the classical risk measure of portfolio variance by the variance of tracking errors between stochastic index return and the return on the portfolio selection. In this report we formulate the index-tracking portfolio optimization model and present an illustrative example where we compare the presented model with the classical Markowitz mean-variance portfolio optimization model.

2 Index-tracking Optimization Model

Consider a market index M and select a portfolio P with n assets. P reproduces the performance of the market index if the return on P, denoted by r_P, follows very closely the return of the index, denoted by r_M, at every unit time period. Thus, the goal is to obtain the P that minimizes the variance of the difference between both returns, r_P and r_M. Denote the return on asset i by r_i and its mean by μ_i, for $i = 1, 2, \ldots, n$. The variance-covariance matrix of assets returns is denoted by V, where σ_{ij} denotes the covariance between the returns of two assets, and σ_{ii} the variance of r_i. The covariance between asset returns and market index returns is given by $\sigma_{iM} = Cov(r_i, r_M)$. The beta of asset i relative to M is given by

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} \tag{1}$$

where σ_M^2 denotes the variance of the market index returns. As previously stated, the index-tracking model aims to minimize $Var(r_P - r_M) = \sigma_P^2 + \sigma_M^2 - 2\sigma_M^2 \beta_P$, the so-called measure of goodness of the index-tracking portfolio. Thus, the optimization problems is formulated as

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follows:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2}x^T V x - \sigma_M^2 \beta^T x \\ \text{subject to} \quad & \sum_{i=1}^{n} \mu_i x_i = m \\ & \sum_{i=1}^{n} x_i = 1 \end{aligned} \tag{2}$$

and $-1 \leq x_i \leq 1$, $i \in \{1, \ldots, n\}$. Therefore, we need to solve a constrained convex quadratic programming problem (QP). Given that the matrix V is symmetric positive semi-definitive and dense, we select the solver `e04ncc`. The following measures are used to determine the characteristics of the optimal portfolio

- Portfolio variance: $\sigma_P^2 = x^T V x$.
- Portfolio beta: $\beta_P = \sum_{i=1}^{n} \beta_i x_i$.
- Portfolio P (measure of goodness): $Var(r_p - r_M) = \sigma_P^2 + \sigma_M^2 - 2\sigma_M^2 \beta_P$.

Note that the tracking-efficient (TE) portfolio is equivalent to Markowitz mean-variance (MV) portfolio when the linear term $-\sigma_M^2 \beta^T x$ is removed from the objective function. A review of the basic Markowitz portfolio optimization theory and solution using the NAG Library can be found in [2].

3 Example

We present a simple example to illustrate the described methodology. We select seven technology stocks in order to track the S&P 500 index. The data for this example was downloaded from Yahoo Finance [3] using a `Python` script with the module `urllib2`. For this example we downloaded monthly prices from January 2009 to January 2016, see Figure 1. From a sample of historical data we can compute the one-period returns for each stock by using the well-known formula

$$r_t = \frac{price_t - price_{t-1}}{price_{t-1}}. \tag{3}$$

Having the returns we can easily determine the mean vector, standard deviation vector, covariance matrix and beta vector, see Table 1 and 2.

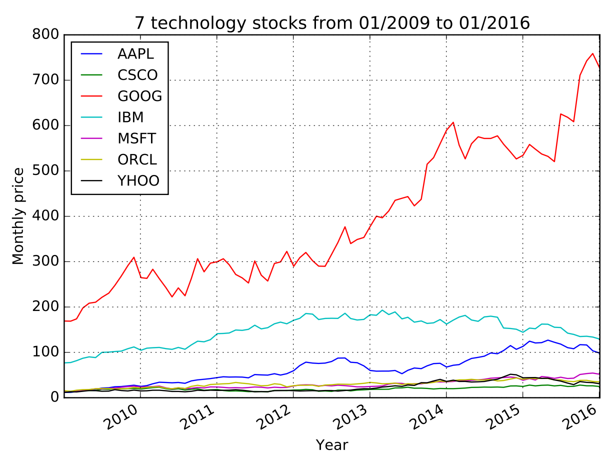

Figure 1: Monthly prices for seven technology stocks from January 2009 to January 2016.

Asset	Mean (μ)(%)	Std. dev (σ)(%)	Beta (β)
ˆGSPC	1.11	4.15	-
AAPL	2.82	7.44	1.026
CSCO	1.08	7.8	1.250
GOOG	2.00	7.3	0.975
IBM	0.72	4.57	0.595
MSFT	1.79	6.78	0.994
ORCL	1.21	6.96	1.199
YHOO	1.49	8.47	0.941

Table 1: Mean, standard deviation and beta.

	AAPL	CSCO	GOOG	IBM	MSFT	ORCL	YHOO
AAPL	0.005528	0.002689	0.001983	0.001417	0.001996	0.002167	0.001418
CSCO	0.002689	0.006082	0.002637	0.001873	0.002812	0.003305	0.002146
GOOG	0.001983	0.002637	0.005324	0.001132	0.002193	0.001718	0.001765
IBM	0.001417	0.001873	0.001132	0.002084	0.001021	0.001571	0.000677
MSFT	0.001996	0.002812	0.002193	0.001021	0.004599	0.002502	0.001350
ORCL	0.002167	0.003305	0.001718	0.001571	0.002502	0.004843	0.002146
YHOO	0.001418	0.002146	0.001765	0.000677	0.001350	0.002146	0.007173

Table 2: Variance-covariance matrix V.

Following the example in [1], we set the desired portfolio mean $m = \mu_M$, where μ_M represents S&P 500 index mean return. Table 3 shows that the index-tracking model, using the market index mean, reduces the number of short positions in the portfolio with respect to Markowitz portfolio. Markowitz portfolio has two short positions (CSCO and ORCL) whereas the index-tracking portfolio has only one short position (AAPL). In addition, the index-tracking portfolio tends to reduce extreme positions, for example IBM changes the long position from 72.1% to a more conservative 44.9%.

Ticker	MV	TE
AAPL	0.019969	-0.023608
CSCO	-0.123901	0.072067
GOOG	0.076037	0.076785
IBM	0.721647	0.449256
MSFT	0.171989	0.115741
ORCL	-0.001755	0.193798
YHOO	0.136014	0.115961

Table 3: Mean-variance and index-tracking optimal portfolios.

Table 4 shows the main portfolio measures. The value of the measure of goodness indicates that the TE portfolio tracks the S&P 500 index more closely with $Var(r_P - r_M) = 0.000707$ compared with the MV portfolio, $Var(r_P - r_M) = 0.001049$.

Measure	MV	TE
Portfolio variance	0.001620	0.001962
Portfolio beta	0.666135	0.864691
Measure of goodness	0.001049	0.000707

Table 4: Portfolio measures.

3.1 Compilation

The presented problem has been solved using the NAG C Library. For example, to compile it with NAG C Library for Linux Mark 26, `CLL6I26DDL`, using the `gcc` compiler follow the instructions below. For other platforms please refer to the appropriate Users' notes of your NAG C Library, see `https://www.nag.co.uk/doc/inun/cl26.html`

Listing 1: Linux instructions.

```
gcc -O3 index_tracking.c -I ../CL26/cll6i26ddl/include/
../CL26/cll6i26ddl/lib/libnagc_nag.so -lpthread -lm -o index_tracking.exe
```

Listing 2: To run the example we pass the downloaded data as standard input.

```
./index_tracking.exe < index_tracking.txt
```

Listing 3: Example data.

```
7 85
GSPC AAPL CSCO GOOG IBM  MSFT ORCL YHOO
1.9E+03  9.8E+01  2.5E+01  7.3E+02  1.3E+02  5.2E+01  3.5E+01 3.1E+01
```

Listing 4: Example output.

```
Optimal portfolio selection:
----------------------------
 GSPC: market index
 AAPL:  -2.36%
 CSCO:   7.21%
 GOOG:   7.68%
  IBM:  44.93%
 MSFT:  11.57%
 ORCL:  19.38%
 YHOO:  11.60%

Portfolio measures:
----------------------------
Portfolio variance : 0.001962
Portfolio beta     : 0.864691
Measure of goodness: 0.000707
```

4 Conclusions

In this tutorial report we have described one of the possible approaches for solving the index-tracking optimization problem. Our approach is based on [1] and the resulting quadratic programming problem has been solved using a solver from Chapter E04 of the NAG C Library. Finally, a detailed description of the obtained results and a comparison to Markowitz mean-variance model was presented.

References

[1] N. C. P. Edirisinghe *Index-tracking optimal portfolio selection*. Quantitative Finance Letters, vol. 1, 16-20, (2013).

[2] NAG Technical report. *Portfolio Optimization using the NAG Library*. (2015). `http://www.nag.co.uk/doc/techrep/pdf/tr1_15.pdf`

[3] Yahoo Finance. Historical stock data. `https://finance.yahoo.com`